RESEARCH STATEMENT
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My research interests are in the overlap of birational geometry, deformation theory, singularity theory and commutative algebra. The main focus of my work is the study of how the volume of a line bundle or its local counterpart change across families of schemes with applications to the deformation theory of singularities. The main result of [Ran1] determines the change of the restricted local volume across flat families for a fairly general class of line bundles. The change turns out to be the degree of certain projective scheme. As a first application of this result I considered the study of singularities that are not necessarily smoothable. I introduced the class of singularities that admit deformations to deficient conormal singularities (DCS); this class contains as a subclass all smoothable singularities. For such singularities the restricted local volume of a particular line bundle provides a relation between algebraic and topological data. Moreover, I showed that the local volume gives a numerical control of Whitney–Thom (differential) equisingularity generalizing previous results about isolated complete intersection singularities.

Yet, it remains unclear which singularities admit deformations to DCS. Also, the generality of my main result suggests it can be applied to other problems that involve families of singularities such as moduli problems, existence of simultaneous resolutions in low dimensions, simultaneous normalizations, and simultaneous canonical models. Another goal will be to understand better for which classes of line bundles appearing naturally in the deformation theory of singularities the restricted local volume is Zariski upper semi-continuous - it’s a problem closely related to determining when the ring of sections of these line bundles are finitely generated.

My other research interests include integral closure methods, and their connection to generalized conormal geometry, singularities, asymptotics of associated primes with applications to symbolic powers.

1. Restricted local volumes and deformation theory

1.1. Restricted local volumes and the main result. The volume of a line bundle is an invariant studied extensively over the past decade and a half by the birational geometers most notably Ein, Cutkosky, Okounkov, Lazarsfeld, Mustata, Popa among others. The use of volume-type invariants to singularity theory was hinted in a recent work by Kleiman, Ulrich and Validashti [KUV] (see also [UV11]) who studied the epsilon multiplicity of a module from the perspective of commutative algebra. The epsilon multiplicity generalizes the Hilbert-Samuel (HS) multiplicity of an ideal primary to the maximal ideal of a local Noetherian ring. In contrast to the usual HS multiplicity, the epsilon multiplicity might be an irrational number as shown by Cutkosky, Herzog and Srinivasan [CHS10]. An important advance achieved in [KUV] is that epsilon multiplicity is an upper semi-continuous over one-parameter deformations; however it’s Zariski upper semi-continuouy has not been established. Moreover, by considering particular examples of deformations of singular curves, Kleiman, Ulrich and Validashti show that the epsilon multiplicity does not have the desirable properties of an invariant for deformations of singularities. The notion of an epsilon multiplicity of an ideal was
generalized by Fulger [Ful13] who introduced the local volume as a local counterpart of the usual volume of a line bundle. Following the work of Ein, Lazarsfeld, Mustata, Nakamaye and Popa [ELMNP], I introduced the notion of restricted local volume and for particular classes of line bundles I studied its properties as an invariant of deformation theory and equisingularity theory.

First, let’s fix some notation. Let $X$ and $Y$ be local schemes of finite type over a field $k$ with closed points $x_0$ and $y_0$ respectively. Assume $x_0$ and $y_0$ have the same residue fields. Suppose $g : X \to Y$ is a flat morphism with equidimensional fibers and a reduced generic fiber. Let $Z$ be an equidimensional projective $\mathcal{O}_X$-scheme such that the structure morphism $c : Z \to X$ maps each irreducible component of $Z$ to an irreducible component of $X$. Set $Z := \text{Proj}(A)$ where $A$ is a graded $\mathcal{O}_X$-algebra. Denote by $r + 1$ the dimension of $Z$. Denote by $[c^{-1}(x_0)]_r$ the dimension $r$ part of the fundamental cycle of $c^{-1}(x_0)$ and by $\deg[c^{-1}(x_0)]_r$ the degree of $[c^{-1}(x_0)]_r$. Finally, let $S$ be a subscheme of $X$ that is finite over $Y$. The main result [Ran1] is the following theorem

**Theorem 1.1** (R.’16). We have

\begin{equation}
\lim_{n \to \infty} \frac{r!}{n^r} \dim_{k(y_0)} H^r_S(A_n) \otimes_{\mathcal{O}_Y} k(y_0) - \lim_{n \to \infty} \frac{r!}{n^r} \dim_{k(y)} H^1_S(A_n) \otimes_{\mathcal{O}_Y} k(y) = \deg[c^{-1}(x_0)]_r.
\end{equation}

For convenience denote by $\text{vol}_A(X_{y_0})$ and $\text{vol}_A(X_y)$ the two limits appearing on the left-hand side of (1). Their existence is guaranteed by recent results of Cutkosky [Cut15] who built upon the work of Kaveh and Khovanskii [KK12] on a generalized Fujita approximation theorem. Both $\text{vol}_A(X_{y_0})$ and $\text{vol}_A(X_y)$ are examples of local volumes in the sense that after suitable compactification procedure where $X$ gets replaced by a projective extension and $Z$ gets replaced by a suitable modification $Z'$, the computation of the two limits reduces suitably to that of the volume of the restrictions on $c^{-1}(X_{y_0})$ and $c^{-1}(X_y)$ of the pullback of $\mathcal{O}_Z(1)$ to $Z'$.

Particular instances of (1) were known before as results of Hironaka and Schickhoff (see Rmk. 2.6 in [Lip82]), Teissier [TS1], Kleiman and Gaffney [GK99], and Gaffney ([Gaff04] and [Gaff08]). For a summary of similar work in the projective setting see [Kol15]. Gaffney’s work on his Multiplicity-Polar Theorem served as my major inspiration for proving (1).

Work over the complex numbers. Set $Z := \text{Bl}_I(X)$ where $I$ is an ideal in $\mathcal{O}_{X,x_0}$ such that $\text{Supp}(\mathcal{O}_{X,x_0}/I)$ is finite over $Y$. Teissier [TS1] proved that the change of the HS multiplicity $c(T_y)$ across $Y$ equals the degree of the top dimensional part of fundamental cycle of irreducible components of $c^{-1}(x_0)$. Next, suppose $M$ is a submodule of a free module $\mathcal{F}$ such that $\text{Supp}(\mathcal{O}_{X,x_0}/\mathcal{F})$ is finite over $Y$. Then Gaffney [Gaff04] proved (1) where in this case $\text{vol}_A(X_{y_0})$ and $\text{vol}_A(X_y)$ are the Buchsbaum–Rim (BR) multiplicities of the pairs $(M_{y_0}, F_{y_0})$ and $(M_y, F_y)$. In fact, Gaffney showed that $\mathcal{F}$ can be substituted with a module that’s not necessarily free, but for which the BR multiplicity is well-defined. In the applications to equisingularity theory $X$ is a family of isolated hypersurface (Teissier), complete-intersection or more generally smoothable singularities (Gaffney) embedded in complex affine space, $\mathcal{M}$ is taken to be the relative Jacobian module, i.e. the module generated by the columns of the Jacobian matrix of partials with respect to the fiber coordinates, and $Z$ is taken to be the relative conormal scheme of $X$ or a modification of it. Then the constancy of the multiplicities is equivalent to the equidimensionality of the fibers of $Z$ which in turn is equivalent to $X$ being a Whitney–Thom equisingular family.
Work in the complex analytic setting. Assume

\[
h: \mathcal{X}, x_0 \to W, w_0
\]

is a deformation of \(X_{y_0}\) such that \(W, w_0\) is an irreducible germ containing \(Y, y_0\) with \(y_0\) being identified with \(w_0\), and the original family \(\mathcal{X}\) is embedded in \(\mathcal{X}'\). Assume \(S_W\) is a subspace of \(\mathcal{X}\) finite over \(W\). In general, it’s not known if \(\text{vol}_A(X_w)\) stabilizes for generic \(w\). Say \(W\) is a \textit{good deformation base space} for \(X_{y_0}\) if there exists a Zariski open dense subset \(U\) of \(W\) such that the restricted local volume \(\text{vol}_A(X_w)\) is constant for \(w \in U\). Apply (1) for \(Y\). Denote by \(y\) the generic point of \(Y\). Now by a conservation of number and a covering argument (c.f. Thm 2.8 in [GR16]) we can replace \(\text{vol}_A(X_{y_0})\) and \(\text{vol}_A(X_y)\), without violating the validity of (1), with the restricted local volumes obtained from deforming \(X_{y_0}\) and \(X_y\) along curves \(W_1\) and \(W_2\) that lie in \(U\). Indeed, apply (1) twice to the one-parameter deformations over \(W_1\) and \(W_2\). Then subtracting one of the equations from the other and taking into account that \(\text{vol}_A(X_{w_1}) = \text{vol}_A(X_{w_2})\), where \(w_1\) and \(w_2\) are the generic points of \(W_1\) and \(W_2\), we get that difference of the replaced volumes is equal to difference of two degrees associated with the deformations over \(W_1\) and \(W_2\).

We claim that this difference of degrees equals to the degree associated with the original family \(X\). This is summarized in the figure below. For the purposes of illustrating the main idea we assume that the fiber of \(Z\) over a point of \(X\) off \(S\) is \(\mathbb{P}^e\) for some \(e\). But \(Z\) is contained in \(\mathcal{X} \times \mathbb{P}^k\) for some \(k\). Intersect \(Z\) with a hyperplane from \(\mathbb{P}^k\) of codimension \(r\). Then by Kleiman’s transversality result the three degrees involved can be realized as sheets of a covering of \(W\) locally off \(y_0\) and \(y\).

Thus for each \(X_{y_0}\) and each \(X_y\) we can associate unique numbers depending on \(W\) and \(S\) such that if \(X_y\) connects to \(X_{y_0}\) as the generic member of a one-parameter deformation of \(X_{y_0}\), then \([c^{-1}(x_0)],\) vanishes if and only if the corresponding restricted volumes associated with \(X_{y_0}\) and \(X_y\) are the same.

Prb. 1 A direction for future research is to generalize in collaboration with Mihai Fulger his local volume construction to the case where the morphism he works with to define the local volume is of arbitrary relative dimension. The next step will be to prove (1) in greater generality. This will allow us to apply (1) to study the deformation theory of singularities appearing in the MMP, existence of equinormalization in the local setting (cf. [CL08]), existence of simultaneous canonical models (cf. [Kol15]), etc.

Prb. 2 A major problem is to determine when the local volumes are Zariski upper semi-continuous at least for the line bundles that appear in the deformation theory of
singularities, i.e. for those line bundles associated with the Jacobian module. Work
with the pullback $L$ of $O_Z(1)$ to $Z'$. For low dimensions like surface singularities we
will try to see under what conditions $L$ is nef or ample using Kleiman’s and Nakai-
Moishezon’s criterions. Recall that if a line bundle $L$ is nef, then by the asymptotic
Riemann–Roch theorem it follows that its volume is the top self intersection number
of the line bundle. So, if the restrictions of $L$ to the fibers of a family remain nef,
then the local volume will be Zariski upper semi-continuous. A related problem is to
determine for which singularities the section ring of the line bundle associated with
the Jacobian module is finitely generated.

1.2. Applications to the deformation theory of singularities. Thm. [1.1] provides a
powerful machinery to study the deformation theory of singularities. Suppose (2) is a
deformation of a germ of an isolated singularity $X_{w_0}$ where $W$ is an irreducible component
of the miniversal base space of $X_{w_0}$. Let $S$ be the singular locus of $X$ and let $Z$ be the relative
conormal scheme or a modification of it. In this case $A$ is the $Rees$ algebra of the Jacobian
module of $X_y$ or a modification of it. In that setup the restricted local volumes provide com-
plete control of the Whitney–Thom equisingularity for families arising from (2) as shown by
the author [Ran1].

**Theorem 1.2.** Let $g: X, x_0 \rightarrow Y, y_0$ be a family of singularities embedded in (2) where $W$ is
a good deformation base space. Then Whitney–Thom equisingularity for $X$ is controlled by
the constancy of $\text{vol}_A(X_y)$.

This result generalizes previous results due to Lê and Teissier for families of isolated hyper-
surface singularities, Gaffney and Kleiman [GK99] for families of isolated complete intersection
singularities, and families of isolated smoothable singularities with smooth miniversal base
space as studied by Gaffney [Gaff04] by means of his multiplicity-polar theorem. In all of
these cases $W$ is smooth and the generic fiber $X_y$ is a smoothing of $X, x_0$.

The point of view where one determines a set of invariants, one for each irreducible compo-
nent of the miniversal base space of $X_{w_0}$, that control the deformations of $X_{w_0}$ was pioneered
for the first time by Gaffney and the author [GR16] for the case of smoothable determinantal
singularities.

The class of isolated singularities which have components of the miniversal base space that
are good deformation base spaces has not been explored yet. It contains as a subclass all
smoothable singularities. Assume $X_{w_0}$ is of pure dimension $d$. Let $Y$ be a smooth curve in $W$
such that $X_y$ is smooth. Combining Cor. 4.8 from [GR16] and Thm. 1.1 we get the following
remarkable Lê-Greuel type formula for smoothable singularities

$$\text{vol}_A(X_{w_0}) = (-1)^d \chi(X_y) + (-1)^{d-1}(X_y \cap H)$$

(3)

that relates restricted local volume to the Euler characteristic, or the Milnor number of $X_{w_0}$
associated with a particular component $W$ of the miniversal base space of $X_{w_0}$, and that of
a generic hyperplane section of a smoothing of $X_{w_0}$. Even more surprisingly, if $X_{w_0}$ is an
arbitrary curve singularity, then $\text{vol}_A(X_{w_0})$ is an intersection number and each component of
the miniversal base space of $X_{w_0}$ is a good deformation base space (see [Ran1]). Furthermore,
if $X_{w_0}$ is smoothable, then

$$\text{vol}_A(X_{w_0}) = \mu + m - 1$$

(4)

where $\mu$ is the Milnor number of $X_{w_0}$ as defined by Buchweitz and Greuel [BG80] and $m$ is
the multiplicity of $X_{w_0}$. 
A more general subclass of isolated singularities which admit a good deformation base space \( W \) is the class of singularities for which the generic deformation \( X_w \) is a deficient conormal singularity (DCS) as introduced in \([\text{Ran1}]\). We say \( X_w \) is DCS if \( X_w \) has deficient conormal scheme at its singular points, or in other words the fibers of the conormal scheme over the singular points are of codimension at least 2. Notably, as shown in \([\text{Ran1}]\), \( \text{vol}_A(X_w) \) vanishes if and only if \( X_w \) is DCS; hence from (1) it follows that \( \text{vol}_A(X_w) \), computed from a generic one-parameter deformation in \( W \), is always an integer. Furthermore, the DCS property is stable under infinitesimal deformations. It can also be shown that many of the examples of rigid singularities considered by Schlessinger \([\text{Sch73}]\) are DCS.

Yet, a third subclass of isolated singularities which admit a good deformation base space \( W \) is the class of surface singularities such that their generic deformation admits a small resolution. One can show that the local volume behaves well under finite maps (cf. \([\text{Ful13}]\)), so we can compute \( \text{vol}_A(X_w) \) as an intersection number on the resolution of \( X_w \). Hence for this class of singularities \( \text{vol}_A(X_w) \) is Zariski upper semi-continuous.

Below I list several directions for future research:

Prb. 1 Our first task is to understand better the class of singularities that admit deformations to DC singularities. Determining whether a singularity is smoothable is in general very hard and elusive problem as discussed by Nemethi \([\text{Nem08}]\) and Hartshorne \([\text{Har10}]\). However, it seems that the class of singularities admitting deformations to DC singularities is much larger by recent work of Gaffney and Ruas \([\text{GRu16}]\), so it’s plausible that one can find a criterion that controls when a given isolated singularity admits a deformation to DCS. Another problem will be to understand the interplay between rigid and DC singularities. It seems that most examples in the work of Schlessinger on rigid singularities are DC, so a reasonable conjecture to work on will be to try to prove that most rigid singularities at least in dimension two or higher are DCS.

The strategy of attack for Prb. 1 will be based on techniques coming from commutative algebra. Concretely, I plan applying integral closure methods in the following way. A way to impose control on the relative conormal space of a generic deformation of a singularity is to use integral closure methods to control the order of vanishing of some of the columns of the relative Jacobian matrix along smooth curves that lie in the total space of the deformation. By the valuative criterion for integral dependence, if these orders for some of the columns happen to be higher than the orders of vanishing of the rest of the columns that will force an integral dependence relation for the former set over the module generated by the latter set of columns. This would imply that the relative Jacobian module can be generated up to integral closure by smaller than expected number of columns. In turn the bound on the number of generators gives a bound for the dimension of the fiber of the relative conormal over a closed points of the underlying space. To understand the interplay between DC and rigid singularities we will make use of the fact that latter class of singularities is characterized by an equality between their Jacobian and normal modules (cf. p. 33 in \([\text{Art76}]\)). Using this relation we will try to construct a regular sequence of elements from the ring of the germ of a rigid singularity of depth at least 2 that is also regular for the Rees algebra of its Jacobian module. That will imply a bound on the codimension of the fiber of the conormal space over the singular point of the rigid singularity.

Prb. 2 We will try to understand better the connection between the local volumes and the topological invariants that appear in \((3)\) and \((4)\). One way to get insight in that
relation is to see what topological information the volume carries for sandwich and quotient singularities. The topology and deformation theory of these classes of singularities are well-understood by the work of Nemethi and collaborators. Another goal will be to find an analogue of (3) for singularities that admit deformations to DCS. Then we will attempt to prove (4) for curves that are not necessarily smoothable. Finally, for singularities admitting deformation to DC we will try to compute \( \text{vol}_A(X_{w_0}) \) algebraically from information encoded by the Jacobian module of \( X_{w_0} \) only.

For this problem I will employ the deformation theoretic methods developed by Nemethi and collaborators to study the topological properties of the local volume on quotient and sandwich singularities. Proving a relation like (3) for the class of singularities that admit deformations to DCS will be a remarkable achievement that will open the door to finding a generalized Milnor number for such singularities and thus generalizing the work of [BG80]. Our first line of attack will be to use (1) along with the attaching topological result of Lê. One way to understand better \( \text{vol}_A(X_{w_0}) \) and relate it to already known topological invariants is to find a down-to-earth algebraic way to compute it. On one hand by (1) we have \( \text{vol}_A(X_{w_0}) = \deg[\text{c}^{-1}(x_0)]_r \). On other hand, the Jacobian module of \( X_{w_0} \) can be embedded into a free module using only those syzygies that can be extended to the total space of the deformation. For such an embedding consider the presentation matrix of the Jacobian module. Then if the total space of the deformation is Cohen-Macaulay, using a conservation of number argument one can try to relate \( \deg[\text{c}^{-1}(x_0)]_r \) to the colength of the ideal of maximal non-vanishing minors of the presentation matrix subtracting the contribution of the singular locus of the deformation.

Prb. 3 A joint project with Terence Gaffney will be to use my Thm. 1.1 to study the deformation theory of nonisolated singularities building on his work with Gassler [GG99] on nonisolated hypersurface singularities. The main idea is to use (1) along with a suitable localization of \( A \) with respect to a component of the singular locus as proposed by Lipman (see Scts. 4 and 5 in [Lip82]).

2. Associated points and Integral Closure of Modules

Following Eisenbud, Huneke and Ulrich [EHU03] define the Rees algebra of \( \mathcal{M} \) as the quotient

\[
\mathcal{R}(\mathcal{M}) := \text{Sym}(\mathcal{M}) / (\cap L_g)
\]

where the intersection is taken over all homomorphisms \( g \) from \( \mathcal{M} \) to a free \( R \)-module \( F_g \) and \( L_g \) denotes the kernel of the induced map \( \text{Sym}(\mathcal{M}) \rightarrow \text{Sym}(F_g) \). In fact as shown in Prp. 1.3 [EHU03] \( \mathcal{R}(\mathcal{M}) \) can be computed from any homomorphism of \( \mathcal{M} \) to a free \( R \)-module whose dual is surjective.

Let \( \mathcal{M} \subset \mathcal{N} \) be a pair of finitely generated \( R \)-modules. Assume that \( \mathcal{M} \) and \( \mathcal{N} \) are either locally free at the generic point of each irreducible component of \( X \), or they are contained in a free \( R \)-module. The inclusion of \( \mathcal{M} \) into \( \mathcal{N} \) induces a map from \( \mathcal{R}(\mathcal{M}) \) to \( \mathcal{R}(\mathcal{N}) \). Denote by \( \mathcal{N}^n \) the \( n \)th homogeneous component of \( \mathcal{R}(\mathcal{N}) \) and by \( \mathcal{M}^n \) the image of the \( n \)th homogeneous component of \( \mathcal{R}(\mathcal{M}) \) in \( \mathcal{R}(\mathcal{N}) \). Finally, define the integral closure \( \overline{\mathcal{M}}^n \) of \( \mathcal{M}^n \) in \( \mathcal{N}^n \) to be the module generated by those elements from \( \mathcal{N}^n \) that satisfy an equation in \( \mathcal{R}(\mathcal{N}) \) of integral dependence over \( \mathcal{R}(\mathcal{M}) \). Our main result Thm. 2.1 is as follows.
Theorem 2.1. Assume $X$ is a local universally catenary Noetherian scheme with closed point $x_0$. Assume that each irreducible component of the fiber of $\text{Proj}(\mathcal{R}(M))$ over $x_0$ is of codimension at most 2. Then for a general element $h$ from the maximal ideal of $\mathcal{O}_{X,x_0}$ we have $h \notin \text{div}(N^n/M^n)$ for each $n$.

First we analyze the set $\bigcup_{n=1}^{\infty} \text{Ass}_X(N^n/M^n)$. It is a hard problem to show that this set is finite, because the modules $M^n$ may not form a finitely generated $R$-algebra. Thm. 2.1 yields a complete classification of the points $x$ of $X$ that appear in this set. They are of two types: generic points of codimension one components of subschemes defined by appropriate Fitting ideals of $N/M$, and generic points of the irreducible components of closed subsets of $X$ where the fiber dimension of the structure morphism $\text{Proj}(M) \to X$ jumps. Then Chevalley’s constructibility result implies that there are finitely many such $x$ and hence $\bigcup_{n=1}^{\infty} \text{Ass}_X(N^n/M^n)$ is a finite set.

More generally, without assuming that $X$ is universally catenary, we prove algebraically that $\text{Ass}_X(N^n/M^n) \subseteq \text{Ass}_X(N^{n+1}/M^{n+1})$ for each $n$, and that the sets $\text{Ass}_X(N^n/M^n)$ and $\text{Ass}_X(N^n/M^n)$ are asymptotically stable as special case of a more general result about finitely generated $R$-algebras. Working in complete generality though, doesn’t yield a satisfactory geometric classification of the points appearing in $\text{Ass}_X(N^n/M^n)$ as in the case when $X$ is universally catenary.

The analogous problems for ideals $I$ were posed by Rees [R56]. Ratliff [Ratl76] proved that if $I$ is an ideal in $R$, then $\bigcup_{n=1}^{\infty} \text{Ass}_X(R/I^n)$ is finite. Then Brodmann [Brod79] inspired by Ratliff’s work showed that $\text{Ass}_X(R/I^n)$ is asymptotically stable. If $\text{ht}(I) \geq 1$ it follows from Ratliff’s Thm. 2.5 [Ratl76] and McAdam and Eakin’s Prp. 5 that $\text{Ass}_X(R/I^n)$ is asymptotically stable. Then Rees [R81] observed that $\bigcup_{n=1}^{\infty} \text{Ass}_X(R/I^n)$ is finite in complete generality as a consequence of his valuation theorem. Finally, Ratliff [Ratl84] proved that $\text{Ass}_X(R/I^n) \subseteq \text{Ass}_X(R/I^{n+1})$ for each $n$ without the assumption $\text{ht}(I) \geq 1$. These results were extended by Katz and Naudé [KaNa95] to the case where $M$ is contained in a free $R$-module $F$ by reduction to the ideal case.

Questions about the asymptotic behavior of prime divisors arise naturally in various problems from commutative algebra including going-down of prime ideals, catenary chain conjectures (see [McA83] and [Ratl83]), symbolic powers of ideals, Swanson’s result on the asymptotics of the primary decomposition of the powers of an ideal [S97] and its applications to symbolic powers (cf. [ELS01]). More recently, the author [Ran1] has found applications to a specialization problem for certain local cohomology modules defined over a flat family of complex analytic spaces, and to characterizing the vanishing of the restricted local volumes that we discussed in Sect. 1.

Assume $X$ is local with closed point $x_0$. McAdam [McA80] showed that if $X$ is formally equidimensional and $x_0 \in \text{Ass}_X(R/I^n)$ for some $n$, then the fiber of $\text{Proj}(R/I)$ over $x_0$ is of minimal codimension. This result was generalized by Katz and Rice [KaR86] using valuation theory to the case where $M$ is contained in a free module $F$ such that $M$ has the same rank as $F$ at the generic point of each irreducible component of $X$. As an immediate consequence of our main result we prove the following far reaching generalization of these results.

Theorem 2.2. Assume $X$ is universally catenary. If $x_0 \in \text{Ass}_X(N^n/M^n)$, then the irreducible components of the fiber of $\text{Proj}(R/M)$ over $x_0$ are of codimension at most one.

The case when $M$ is contained in a free $R$-module $F$ is of particular interest. In this case we prove a converse to Thm. 2.2 assuming only that $X$ is a Noetherian local scheme.
Theorem 2.3. If the irreducible components of the fiber of $\text{Proj}(\mathcal{R}(\mathcal{M}))$ over $x_0$ are of codimension at most one, then $x_0 \in \text{Ass}_X(\mathcal{F}^n/\mathcal{M}^n)$.

As a corollary we prove a result which yields a complete classification of the points of $\text{Ass}_X(\mathcal{F}^n/\mathcal{M}^n)$ for arbitrary affine Noetherian scheme $X$. This result generalizes results due to Burch [Bur68] for ideals, and Rees [R87], Katz and Rice (see Thm. 3.5.1 in [KaR08]) for the case where the rank of $\mathcal{M}$ at the generic point of each irreducible component of $X$ is equal to $\text{rk}(\mathcal{F})$.

As another application of our main result, we recover, strengthen, and prove a sort of converse to an important result due to Kleiman and Thorup about integral dependence of modules. Assuming that $X$ is equidimensional and universally catenary, they show (see [KT94] for the original proof and [KT01] for a much shorter and more elementary proof) that the inverse image in $\text{Proj}(\mathcal{R}(\mathcal{M}))$ of the locus in $X$ where $N$ is not integral over $M$ is of codimension one. Their result has found numerous applications to equisingularity theory (c.f. [GK99], [Gaf08], and [GR16]). It is used as a crucial step in establishing that if a given equisingularity condition holds generically, then it holds everywhere.

The original proof of Kleiman and Thorup involves an analysis of an intermediate object: the exceptional divisor of the blowup of $\text{Proj}(\mathcal{R}(\mathcal{N}))$ with center the ideal $\mathcal{M}\mathcal{R}(\mathcal{N})$. In our treatment, we work directly on $\text{Proj}(\mathcal{R}(\mathcal{M}))$. In this sense, our approach is similar in spirit to that of Simis, Ulrich and Vasconcelos [SUV01], who analyze the integrality of $\mathcal{R}(\mathcal{N})$ over $\mathcal{R}(\mathcal{M})$ by reducing it to a local problem at codimension one primes of $\mathcal{R}(\mathcal{M})$. However, in contrast to [SUV01], [KT94] and [KT00], our approach allows us to work in a more general setting: we don’t require $X$ to be equidimensional, and we allow the ranks of $M$ and $N$ to vary across the generic points of the irreducible components of $X$. Finally, we strengthen the Kleiman–Thorup theorem by showing that the inverse image in $\text{Proj}(\mathcal{R}(\mathcal{M}))$ of each irreducible component of the locus in $X$ where $N$ is not integral over $M$ is of codimension one in an irreducible component of $\text{Proj}(\mathcal{R}(\mathcal{M}))$.

Most of the techniques and results of [Ran2] extend to a fairly general setting - we can replace $\mathcal{R}(\mathcal{M})$ and $\mathcal{R}(\mathcal{N})$ by two finitely generated graded $R$-algebras $A \subset B$ as noted at the end of Sct. 4 in [Ran2].

References


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