Representations of quivers and their applications.

3 lectures by

Jerzy Weyman

1. The basics of quiver representations.

The theory of quiver representations took off in early seventies after Gabriel Theorem indicated deep connections with representations of Lie algebras and root systems. The principal results were obtained by Gabriel, Kac, Auslander, Reiten, Ringel, Schofield and Crawley-Boevey.

**Definition 1.1.** A quiver $Q$ is a pair $Q = (Q_0, Q_1)$ consisting of the finite set of vertices $Q_0$ and the finite set of arrows $Q_1$. The set $Q_0$ usually will be identified with the set $[1,n] = \{1,\ldots,n\}$. The arrows will be denoted by initial letters of the alphabet. Each arrow $a$ has its head $ha$ and tail $ta$, both in $Q_0$:

$$ta \xrightarrow{a} ha$$

Formally both $h$ and $t$ are just maps from $Q_1$ to $Q_0$.

We fix an algebraically closed field $K$.

**Definition 1.2.** A (finite dimensional) representation $V$ of $Q$ is a family of finite dimensional $K$-vector spaces \{ $V(x); x \in Q_0$ \} and of $K$-linear maps $V(a) : V(ta) \to V(ha)$. A morphism $f : V \to V'$ of two representations is a collection of $K$-linear maps $f(x) : V(x) \to V'(x); x \in Q_0$, such that for each $a \in Q_1$ we have $f(ta)V'(a) = V(a)f(ha)$. This in fact means that the following diagrams commute

$$
\begin{array}{ccc}
V(ta) & \xrightarrow{V(a)} & V(ha) \\
\downarrow f(ta) & & \downarrow f(ha) \\
V'(ta) & \xrightarrow{V'(a)} & V'(ha)
\end{array}
$$

We denote the linear space of morphisms from $V$ to $V'$ by $Hom_Q(V,V')$. Obviously the representations of a quiver $Q$ over $K$ form a category denoted $Rep_K(Q)$.

Let us consider some basic examples illustrating the meaning of representations.
Example 1.3. The quiver $A^e_2$ is $1 \overset{a}{\longrightarrow} 2$. The category $\text{Rep}_K(\theta(1))$ can be identified with the category of linear maps $V(1) \overset{V(a)}{\longrightarrow} V(2)$. If $V(a) : V(1) \rightarrow V(2)$ and $W(a) : W(1) \rightarrow W(2)$ are two linear maps then their morphism is a pair of linear maps $f(i) : V(i) \rightarrow W(i)$, $i = 1, 2$, such that the diagram

\[
\begin{array}{ccc}
V(1) & \overset{V(a)}{\longrightarrow} & V(2) \\
f(1) \downarrow & & \downarrow f(2) \\
W(1) & \overset{W(a)}{\longrightarrow} & W(2)
\end{array}
\]

commutes.

Example 1.4. The one loop quiver $\sigma(1)$ with $\sigma(1)_0 = \{1\}, \sigma(1)_1 = \{a\}$ with $ta = ha = 1$. The category $\text{Rep}_K(Q)$ can be identified with the category of endomorphisms of $K$-vectorspaces.

Example 1.5. More generally, the $m$-loop quiver $\sigma(m)$ has one vertex, $\sigma(m)_0 = \{1\}$ and $m$ arrows $a_1, \ldots, a_m$ with $ta_i = ha_i = 1$.

Example 1.6. The $m$-subspace quiver $\xi(m)$ has $m + 1$ vertices, $\xi(m)_0 = \{1, 2, \ldots, m + 1\}$ and with $m$ arrows $a_1, \ldots, a_m$ with $ta_i = i, ha_i = m + 1$.

Example 1.7. The equioriented $A_n$ quiver $A^e_n$

\[A^e_n : 1 \rightarrow 2 \rightarrow 3 \rightarrow \ldots \rightarrow n - 1 \rightarrow n.\]

We denote the arrow $i \rightarrow (i + 1)$ by $a_i$.

In order to obtain first properties of categories $\text{Rep}_K(Q)$ we identify them as module categories. We start with some preparatory definitions.

A path $p$ in $Q$ is a sequence of arrows $p = a_1, \ldots, a_m$ such that $ha_i = ta_{i+1}$ ($1 \leq i \leq m - 1$). We define $tp = ta_1, hp = ha_m$. For each $x \in Q_0$ we also include in our definition trivial path $e(x)$ from $x$ to $x$. For $x, y \in Q_0$ we define $[x, y]$ to be the $K$-vector space on the basis of paths from $x$ to $y$.

Definition 1.7. The path algebra $KQ$ of a quiver $Q$ is a $K$-algebra with a basis labeled by paths in $Q$. We denote by $[p]$ the element in $KQ$ corresponding to the path $p$ in $Q$. We define the multiplication by the formula

\[
[p] \circ [q] = \begin{cases} [pq], & \text{if } tp = hq; \\ 0, & \text{otherwise.} \end{cases}
\]

If $p$ is a path with $tp = x, hp = y$, this includes $p \circ e(x) = p, e(y) \circ p = p$.

Notice that the trivial paths $e(x)$ ($x \in Q_0$) form an orthogonal set of idempotents in $Q$. 

2
Example 1.3.a). The algebra $K\theta(1)$ is 3-dimensional over $K$. The basis consists of $[e(1)], [e(2)], [\alpha]$ with the multiplication table given by formulas $[e(i)]^2 = [e(i)]$ for $i = 1, 2$, $[e(1)]e(2) = [e(2)]e(1) = 0$, $[\alpha] = [\alpha(2)][\alpha] = [\alpha][e(1)]$, $[\alpha][e(2)] = [e(1)][\alpha] = 0$.

Example 1.4.a). The algebra $K\sigma(1)$ is a polynomial ring in one variable over $K$, generated by $[\alpha]$.

Example 1.5.a). The algebra $K\sigma(m)$ is a free polynomial algebra $K\prec [\alpha_1], \ldots, [\alpha_m] \succ$ on $m$ noncommuting variables.

Proposition 1.8. The categories $\text{Rep}_K(Q)$ and $KQ - \text{Mod}$ are equivalent.

Proof. Let us define two functors giving the equivalences. The functor $F : KQ - \text{Mod} \to \text{Rep}_K(Q)$ is defined as follows. For a $KQ$-module $M$ we define the vector spaces $FM(x) := e(x)M$ for $x \in Q_0$. It is clear that if $a \in Q_1$ then the multiplication by $a$ sends $FM(ta)$ into $FM(ha)$ because $ac(ta) = e(ha)a$. So the set of spaces $FM(x)$ and of linear maps $FM(a) : FM(ta) \to FM(ha)$ given by the action of $a$ define a representation $FM$ of $Q$.

To define the inverse $G : \text{Rep}_K(V) \to KQ - \text{Mod}$ we notice that we can take $GV = \oplus_{x \in Q_0} V(x)$ as a vector space and we can define the left action of $a$ to be zero on all spaces $V(x)$ with $x \neq ta$, and to be applying $V(a)$ on $V(ta)$. It is easy to check that this indeed defines a $KQ$-module $GV$ and that the functors $F, G$ define the equivalence of categories.

Throughout these notes we will not distinguish between the representations of $Q$ and $KQ$-modules.

Definition 1.9. Representation $V$ is indecomposable if it cannot be expressed as a proper direct sum, i.e. from the decomposition $V = V' \oplus V''$ it follows that either $V'$ or $V''$ is zero.

Corollary 1.10.(Krull-Remak-Schmidt Theorem). Every finite dimensional representation $V$ of a quiver $Q$ is isomorphic to a direct sum of indecomposable representations. This decomposition is unique up to isomorphism and permutation of factors. More precisely, if

$$V \cong V_1 \oplus \ldots \oplus V_m = W_1 \oplus \ldots \oplus W_p$$

with $V_i, W_j$ nonzero and indecomposable, then $m = p$ and there exists a permutation $\sigma$ on $m$ letters such that for each $i, 1 \leq i \leq m$ we have $V_i \cong W_{\sigma(i)}$.

The dimension vector of a representation $V$ is the function $d(V)$ defined by $d(V)(x) := \dim V(x)$. The dimension vectors lie in the space $\Gamma(Q) = \mathbb{Z}^{Q_0}$ of integer valued functions on $Q_0$. The dimension vectors correspond to a sublattice $\Gamma^+(Q)$ in $\Gamma(Q)$ of functions with nonnegative values.

We can look at a problem of classifying representations of the quiver $Q$ of dimension vector $\alpha \in \Gamma^+(Q)$ as a problem of classifying orbits of a group action. More precisely tha
space $\text{Rep}_K(Q, \alpha)$ of representations of $Q$ of dimension vector $\alpha$ can be identified with the affine space

$$\text{Rep}_K(Q, \alpha) = \prod_{a \in Q_1} \text{Hom}_K(K^{\alpha(ta)}, K^{\alpha(ha)}).$$

The product of general linear groups $\text{GL}(Q, \alpha) = \prod_{x \in Q_0} \text{GL}(\alpha(x), K)$ acts on $\text{Rep}_K(Q, \alpha)$ in a natural way (by changes of bases in spaces $K^{\alpha(x)}$). It is clear that two representations $V, V' \in \text{Rep}_K(Q, \alpha)$ are isomorphic if and only if they belong to the same orbit of this action.

Sometimes we will use more equivariant notation. We will fix the vector spaces $V(x)$ of dimensions $\alpha(x)$ for $x \in Q_0$, and then

$$\text{Rep}_K(Q, \alpha) = \prod_{a \in Q_1} \text{Hom}_K(V(ta), V(ha)), \text{GL}(Q, \alpha) = \prod_{x \in Q_0} \text{GL}(V(x)).$$

Ideally the representation theory should give the classification of all isomorphism classes of representations of a quiver $Q$. This is possible for some simple quivers.

**Example 1.3b).** The classification of isomorphism classes of linear maps amounts to classifying the $m \times n$ matrices up to row and columns operations. It is known from the basic linear algebra that the only invariant of matrices is rank. Let us look at this classification from the point of view of representations. The rank classification means that every representation of dimension vector $(m,n)$ is a direct sum of three types of indecomposable representations: $\epsilon_1 : K \rightarrow 0$, $\epsilon_2 : 0 \rightarrow K$ and $\epsilon_{1,2} : K \rightarrow K$ (with the identity map). The $m \times n$ matrix has rank $r$ when the summand $\epsilon_{1,2}$ occurs $r$ times in its decomposition.

**Example 1.4b).** The classification of isomorphism classes is equivalent to the classification of endomorphisms of a vector space up to conjugation. This is a Jordan classification. In terms of indecomposable objects this means that in dimension vector $(n)$ there is a one parameter family of indecomposable endomorphisms $J_n(\lambda)$ given by an $n \times n$ matrix

$$J_n(\lambda) = \begin{pmatrix}
\lambda & 1 & 0 & \ldots & 0 & 0 \\
0 & \lambda & 1 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & \lambda & 1 \\
0 & 0 & 0 & \ldots & 0 & \lambda
\end{pmatrix}.$$  

**Example 1.5b).** The classification of isomorphism classes is the classification of $m$-tuple of endomorphisms of a vector space up to simultaneous conjugation. This is a well known “wild” problem which seems to be very difficult.

**Example 1.6b).** The representation of $m$-subspace quiver $\xi(m)$ with injective arrows gives an $m$-tuple of subspaces $V(a_i)(V(i))$ in the vector space $V(m+1)$, for $i = 1, \ldots , m$. 

4
The classification of indecomposable representations with injective arrows amounts to classifying the \( m \)-tuples of subspaces up to simultaneous change of basis.

**Example 1.7b.** A representation of \( A_n^e \) is a sequence of linear maps

\[
V(1) \rightarrow V(2) \rightarrow V(3) \rightarrow \ldots \rightarrow V(n-1) \rightarrow V(n).
\]

A path \( p = a_1 \ldots a_n \) is an oriented cycle if \( ha_1 = ta_n \). Sometimes we will be making an assumption that the quiver \( Q \) has no oriented cycles.

**Proposition 1.11.** Quiver \( Q \) has no oriented cycles if and only if the algebra \( KQ \) is finite dimensional over \( K \).

The basic result in theory of quiver representations is Gabriel Theorem classifying the quivers \( Q \) of finite representation type.

**Definition 1.12.** A quiver \( Q \) has finite representation type if the category \( \text{Rep}_K(Q) \) contains finitely many isomorphism classes of indecomposable objects. For a quiver \( Q \) we define the graph \( \Gamma(Q) \) to be the graph we get when we forget the orientation of arrows in \( Q \).

**Theorem 1.13 (Gabriel, 1973).** A quiver \( Q \) has finite representation type if and only if the graph \( \Gamma(Q) \) is a Dynkin quiver, i.e. it is on the following list.

- \( A_n \): \( x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_n \)
- \( D_n \): \( x_1 \rightarrow x_2 \rightarrow \ldots \rightarrow x_{n-2} \rightarrow x_{n-1} \)
- \( E_6 \): \( x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_5 \)
- \( E_7 \): \( x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_5 \rightarrow x_6 \)
- \( E_8 \): \( x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_4 \rightarrow x_5 \rightarrow x_6 \rightarrow x_7 \)

In particular the representation finite type property does not depend on the orientation of the arrows.

In fact Gabriel result gives more detailed information about the indecomposable representations for Dynkin quivers. Actually if \( Q \) is a Dynkin quiver Gabriel also proved that
the number of indecomposable representations is equal to the number of positive roots, and their dimension vectors are the positive roots written in terms of simple roots. Thus even the dimension vectors of indecomposables do not depend on the orientation.

Let us look at some examples.

**Example 1.7c).** Let $Q = A_n$. Denote by $e_i$ the dimension vector which is 1 at the vertex $i$ and 0 otherwise. For each $1 \leq i \leq j \leq n$ define the representation $E_{i,j}$ with dimension vector $\epsilon_{i,j} = e_i + e_{i+1} + \ldots + e_j$ by setting the maps between one dimensional vector spaces to be the identity maps. Thus $E_{i,j}$ is the representation

$$
\ldots 0 \rightarrow K \rightarrow K \rightarrow \ldots \rightarrow K \rightarrow 0 \rightarrow \ldots
$$

$i$  $i+1$  $j$

There are $\binom{n+1}{2}$ of such representations. Looking at any book in Lie algebras you will see that the simple positive roots for type $A_n$ are the roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$ and positive roots are $\epsilon_{i,j} = \epsilon_i - \epsilon_j$. Thus $\epsilon_{i,j} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j$. So these are all indecomposables.

**Example 1.6c).** Let us look at the 3-subspace quiver $\xi(3)$. This is a quiver of type $D_4$. If you look in the book on Lie algebras you will see that the root system of type $D_4$ has 12 positive roots. So let’s try to find out the 12 indecomposable representations. Here they are: all connected subgraphs of $\Gamma(Q)$ give us an indecomposable representation with the dimension vector being the characterisitc function of this subgraph (with the identity maps between one dimensional spaces everywhere). This gives 4 representations for subgraph with one vertex, 3 representations for subgraphs with two vertices, 3 representations for subgraph with three vertices and one representation for subgraph with 4 vertices. This gives $4 + 3 + 3 + 1 = 11$ indecomposable representations. The remaining one (after looking onto a book on root systems) has to have dimension vector

$$
\begin{pmatrix}
1 & 2 & 1 \\
1
\end{pmatrix}
$$

Thus we can take the representation

$$
K \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} K^2 \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} K
$$

$\uparrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$K$

to be the twelfth indecomposable.

Gabriel Theorem was generalized to arbitrary quivers by Kac ([K1], [K2], 1980), who proved that indecomposable representations occur in dimension vectors that are the roots
of so-called Kac-Moody Lie algebra associated to a given graph. In particular, these
dimension vectors do not depend on the orientation of the arrows in \( Q \).

Of particular interest are so-called tame quivers. For these in each dimension vec-
tor there is at most one parameter family of indecomposable representations. Nazarova
and Roiter (see also Donovan and Freislich (1976)) showed that the quiver \( Q \) is of tame
representation type if and only if \( \Gamma(Q) \) is the extended Dynkin quiver.

**Remarks on the proof of Gabriel Theorem.**

It is not difficult to show that if \( Q \) is of finite representation type, then \( \Gamma(Q) \) is
Dynkin. In fact, suppose this is true. Consider a dimension vector \( \alpha \) and the action of
the group \( GL(Q, \alpha) \) on \( \text{Rep}_K(Q, \alpha) \). If \( \text{Rep}_K(Q) \) has finitely many isomorphism classes of
indecomposable objects then this action has finitely any orbits. Thus one of these orbits
has to be open, so in particular

\[
\dim GL_K(Q, \alpha) \geq \dim \text{Rep}_K(Q, \alpha).
\]

In fact the inequality is sharp since the scalar subgroup consisting of the homothety by the
same scalar at each vertex takes every representation to an isomorphic one. We introduce
a quadratic form (so-called Tits form)

\[
T_Q(\alpha) = \sum_{x \in Q_0} \alpha^2(x) - \sum_{a \in Q_1} \alpha(ta) \alpha(ha).
\]

Notice that \( T_Q \) does not depend on the orientation of \( Q \), so in fact it depends only on
\( \Gamma(Q) \). We might denote it by \( T_\Gamma \).

Thus if \( Q \) is of finite representation type then the quadratic form \( T_\Gamma(Q) \) has to be
positive definite. Looking into Humphreys book on Lie algebras ([Hu]) we see that this
property characterizes Dynkin graphs (in the context of Lie algebras the quadratic form
comes from the Killing form).

Thus we established that if \( Q \) is of finite representation type, then it has to be Dynkin.

The proof that every Dynkin quiver is of finite representation type is best understood
in terms of Auslander-Reiten techniques. Let us consider the case of equioriented quiver
\( A_n^{eq} \) (Example 1.7). We know that indecomposable representations are the representations
\( E_{i,j} \) for \( 1 \leq i \leq j \leq n \). Let us investigate the maps between these representations. It is
clear that there is a non-zero map \( f: E_{i,j} \to E_{k,l} \) if and only if \( k \leq i \leq l \leq j \), and its image
is \( E_{i,l} \). Thus every map which is not an isomorphism can be written as a composition of
injections \( E_{i,j} \to E_{i-1,j} \) or surjections \( E_{i,j} \to E_{i,j-1} \). These are so-called irreducible maps
between indecomposable modules. They can be fit into the so-called Auslander-Reiten
quiver, which, in case of \( A_4^{eq} \) looks like this.
This AR quiver has the following features:

- Each square $E_{i,j} \rightarrow E_{i-1,j} \oplus E_{i,j-1} \rightarrow E_{i-1,j-1}$ and triangles $E_{i,i} \rightarrow E_{i-1,i} \rightarrow E_{i-1,i-1}$ give exact sequences of representations (almost split sequences),

- There is a translation $\tau$ sending $E_{i,j}$ to $E_{i+1,j+1}$ which preserves the structure of the AR quiver except at the ends (it takes value zero at the modules $E_{i,n}$). In fact the translation $\tau$ can be defined as a functor on $\text{Rep}_K(Q)$. The modules $E_{i,n}$ are the indecomposable projective representations (see below). We also have the translation $\tau^{-1}$ taking $E_{i,j}$ to $E_{i-1,j-1}$. This takes value zero at the modules $E_{1,i}$ (the injective indecomposables),

- The left end of the almost split sequence having $E_{i,j}$ at the right end is $\tau(E_{i,j})$. The right end of the almost split sequence having $E_{i,j}$ at the left end is $\tau^{-1}(E_{i,j})$.

The AR quiver allows to construct the indecomposables systematically. We start with the indecomposable injectives, which correspond to the vertices of the quiver $Q$ (with irreducible maps between them corresponding to arrows in $Q$) and we apply the translation functor $\tau$ repeatedly to get all the indecomposables.

**Example 1.6.c.** Here are the AR quivers of the 2 and 3-subspace quivers $\xi(2)$ and $\xi(3)$. Instead of indecomposables we write their dimension vectors

\[
\xi(2) : 1 \rightarrow 3 \leftarrow 2.
\]

\[
\begin{array}{ccc}
110 & 001 \\
\uparrow & \downarrow & \uparrow \\
010 & 111 \\
\downarrow & \uparrow & \downarrow \\
011 & 100
\end{array}
\]

\[
AR(\xi(2)) : \begin{array}{ccc}
1 & 4 & 2 \\
\uparrow & \downarrow & \downarrow \\
\uparrow & 3
\end{array}
\]

The quiver $AR(\xi(3))$ is

8
Again the translation functors $\tau$ and $\tau^{-1}$ are just the right and left translation in the AR graph.

2. The Klyachko inequalities and quiver representations.

Let us start with four problems seemingly unrelated to each other.

1) Let $A, B, C$ be abelian $p$-groups. We can associate to $A, B, C$ three partitions $\lambda, \mu, \nu$ such that

$$A = \bigoplus_i \mathbb{Z}/p^{\lambda_i} \mathbb{Z}, B = \bigoplus_i \mathbb{Z}/p^{\mu_i} \mathbb{Z}, C = \bigoplus_i \mathbb{Z}/p^{\nu_i} \mathbb{Z}$$

with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$, $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_n$. The question is: what conditions for $\lambda, \mu, \nu$ need to be satisfied to have an exact sequence

$$0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0.$$

In fact one might introduce the coefficient $e_{\lambda, \mu}^\nu$ which is the number of subgroups of type $\mu$ inside of an abelian group $C$ of type $\nu$ with factor of type $\lambda$. Then the question becomes when do we have $e_{\lambda, \mu}^\nu > 0$.

2. Let $A, B, C$ be $n \times n$ Hermitian matrices. They have real eigenvalues that can be arranged in non-increasing order. Let eigenvalues of $A$ be $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, the eigenvalues of $B$ be $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$ and let the eigenvalues of $C$ be $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_n$. What are the conditions for $\lambda, \mu, \nu$ when $C = A + B$?
3. Irreducible representations of $SL(n\mathbb{C})$ are parametrized by the highest weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0$. For which $\lambda, \mu, \nu$ the representation $V(\nu)$ occurs in the tensor product $V(\lambda) \otimes V(\mu)$. In fact here again we can introduce the coefficients $c_{\lambda,\mu}^\nu$ which are the multiplicities of $V(\nu)$ in $V(\lambda) \otimes V(\mu)$. Then again the question is when $c_{\lambda,\mu}^\nu > 0$?

4. Schubert calculus. Let $X = Grass(n, \mathbb{C}^m)$. This is a complex manifold of dimension $n(m-n)$. Let us fix a basis $\{e_1, \ldots, e_m\}$ of $\mathbb{C}^m$ and let us fix a flag 

$$0 = F_0 \subset F_1 \subset \ldots \subset F_{m-1} \subset F_m = \mathbb{C}^m$$

where $F_i$ is the span of $e_1, \ldots, e_i$. Now, for a sequence of numbers $P = (p_1, \ldots, p_n)$ such that $1 \leq p_1 < p_2 < \ldots < p_n \leq m$ we denote

$$\Omega_P(F) = \{L \in X \mid \dim(L \cap F_{p_i}) \geq i \text{ for } i = 1, \ldots, n\}.$$

The subset $\Omega_P(F)$ is closed, irreducible, of dimension $\sum_{i=1}^n (p_i - i)$. For a partition $\lambda$ such that $m-n \geq \lambda_1 \geq \ldots \geq \lambda_n \geq 0$ we denote by $|\lambda| = \sum_i \lambda_i$. We define $\sigma_\lambda \in H^{2|\lambda|}(X)$ the cohomology class whose cap product with the fundamental class of $X$ gives $\omega_P = [\Omega_P(F)]$, where $p_i = m-n+i-p_i$ for $i = 1, \ldots, n$. Thus identify homology and cohomology via Poincare duality we have

$$\sigma_\lambda = \omega_P$$

where $\lambda_i = m-n+i-p_i$ ($1 \leq i \leq n$).

The classes $\sigma_\lambda$ give a $\mathbb{Z}$-basis of $H^*(X)$ so we have (the nonnegative) coefficients $d^\nu_{\lambda,\mu}$ such that

$$\sigma_\lambda \sigma_\mu = \sum_{\nu} d^\nu_{\lambda,\mu} \sigma_{\nu}.$$

again we can ask when $d^\nu_{\lambda,\mu} > 0$.

The link between the four problems is that they all have the same solution. In fact the link between 1), 3) and 4) was known some time ago, because the coefficients $c_{\lambda,\mu}^\nu$, $d^\nu_{\lambda,\mu}$, $e_{\lambda,\mu}^\nu$ are all equal. They are given by the Litllewood-Richardson rule. In particular they are zero unless $|\nu| = |\lambda| + |\mu|$.

The additional conditions were given conjecturally by Horn for 2). They are in the form of inequalities

$$\sum_{k \in K} \nu_k \leq \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j \quad (*)_{I,J,K}$$

where the subsets $I, J, K$ of $[1, n]$ have the same cardinality $r$ (which can take any value between 1 and $n$). See [F] for the history of these inequalities.
Example 2.1. For \( n = 2 \) the inequalities are

\[
\nu_1 \leq \lambda_1 + \mu_1, \nu_2 \leq \lambda_1 + \mu_2, \nu_2 \leq \lambda_2 + \mu_1
\]

together with equality \( \nu_1 + \nu_2 = \lambda_1 + \lambda_2 + \mu_1 + \mu_2 \).

It remains to determine the set \( T^n_r \) of triples \((I, J, K)\) of subsets of cardinality \( r \) in \([1, n]\) which give needed inequalities.

Horn’s conjecture was formulated as follows. One defines the set \( U^n_r \) of triples \((I, J, K)\)

\[
U^n_r = \{ (I, J, K), I, J, K \subset [1, n], |I| = |J| = |K| = r, \sum_{i \in I} i + \sum_{j \in J} j = \sum_{k \in K} k + \frac{r(r+1)}{2} \}.
\]

Then the set \( T^n_r \) can be defined recursively as follows.

a) If \( r = 1 \) we have \( T^n_1 = U^n_1 \),

b) \( T^n_r = \{ (I, J, K) \in U^n_r | \forall p < r, \forall (F, G, H) \in T^n_p, \sum_{f \in F} i_f + \sum_{g \in G} j_g \leq \sum_{h \in H} k_h + \frac{p(p+1)}{2} \} \).

Horn conjectured in ([H], 1962) that the set of inequalities \((\ast)_{(I, J, K)}\) with \((I, J, K) \in T^n_r\) gives (together with equality \( |\nu| = |\lambda| + |\mu| \)) the solution of \( 2 \).

Klyachko in ([K], 1998) proved the Horn conjecture by linking it to \( 1), 3 ) 4 \) (in fact to \( 3 )) and using the fact that the symplectic geometry quotients are the same as geometric invariant theory quotients. See [F] for the details. However, the correspondence was not perfect. Klyachko proved in fact that

\[
(I, J, K) \in T^n_r \iff \exists N \in \mathbb{Z}^+ c^{N\nu}_{N\lambda, N\mu} > 0.
\]

Therefore in case 3) the problem that remained was whether the set of triples \((\lambda, \mu, \nu)\) such that \( c^{\nu}_{\lambda, \mu} > 0 \) is saturated, i.e. whether \( c^{N\nu}_{N\lambda, N\mu} > 0 \) for some \( N \) implies \( c^{\nu}_{\lambda, \mu} > 0 \). Surprisingly the Littlewood-Richardson rule is not very helpful in proving such result. The saturation was proved by Knutson and Tao in ([KT], 1999) by combinatorial argument. For the remainder of this lecture we discuss how this result is a special case of a general result on semi-invariants of quiver representations.

Let \( Q = (Q_0, Q_1) \) be a quiver. We assume that \( Q \) has no oriented cycles, i.e. paths with the same head and tail. We briefly investigate the structure of projective and injective resolutions in \( \text{Rep}_K(Q) \).

First of all let us notice that the simple representations are in a bijection with vertices \( Q_0 \). The simple representation \( S_x \) is just the representation for which \( S_x(x) \) is one...
dimensional and $S_x(y) = 0$ for $x \neq y$. For each vertex $x \in Q_0$ we define an indecomposable projective representation $P_x$ as follows. Let $[x, y]$ denotes a vector space over $K$ with a basis labelled by all paths from $x$ to $y$ in $Q$. We denote by $[p]$ the basis element in $[x, y]$ corresponding to the path $p$.

$$P_x(y) = [x, y], P_x(a) = a \circ [x, ta] \rightarrow [x, ha]$$

Here $a \circ$ denotes a linear map taking the basis element $[p]$ to $[a \circ p]$.

**Proposition 2.2.** The representations $P_x$ are indecomposable, projective. More precisely, there is a functorial isomorphism

$$\text{Hom}_Q(P_x, V) = V(x),$$

and therefore the functor $\text{Hom}_Q(P_x)$ is exact. Every indecomposable projective representation is isomorphic to $P_x$ for $x \in Q_0$.

**Proof.** The equivalence $\text{Hom}_Q(P_x, V) = V(x)$ takes a morphism $\phi$ to $\phi([e(x)])$. The point is that for every $v \in V(x)$ there exists unique homomorphism $\phi_v : P_x \rightarrow V$ such that $\phi_v([e(x)]) = v$. Indeed, we define $\phi([p]) := V(p)v$. To prove that every indecomposable projective module is isomorphic to some $P_x$ we recall that the module is projective if and only if it is a direct summand of a free module. But considering $KQ$ as a left module over itself, we notice that $KQ = \bigoplus_{x \in Q_0} P_x$ where we identify $P_x$ with $KQe_x$. Then the result follows from Corollary 1.11.●

Similarly we define the the indecomposable injective modules $Q_x$. We set

$$Q_x(y) = [y, x]^*, Q_x(a) = (a)^* : [ta, x]^* \rightarrow [ha, x]^*$$

where $^*$ denotes the dual space, and $a : [ha, x] \rightarrow [ta, x]$ is a linear map taking basis element $[p]$ into $[p \circ a]$.

**Proposition 2.3.** The representations $Q_x$ are indecomposable, injective. More precisely, there is a functorial isomorphism

$$\text{Hom}_Q(V, Q_x) = V(x)^*,$$

and therefore the functor $\text{Hom}_Q(-, Q_x)$ is exact. Every indecomposable projective representation is isomorphic to $Q_x$ for $x \in Q_0$.

**Exercise 2.3.** Let $S_x$ be a simple module corresponding to a vertex $x \in Q_0$. There is a natural map $p_x : P_x \rightarrow S_x$ sending the generator $[e_x]$ to the element $1 \in K = S_x(x)$. Prove that the kernel of this map is $\bigoplus_{a \in Q_1; ta = x} P_{ha}$. The resulting complex

$$0 \rightarrow \bigoplus_{a \in Q_1; ta = x} P_{ha} \rightarrow P_x \rightarrow S_x \rightarrow 0$$
gives a projective resolution of length 1 of a simple object $S_x$.

This suggests that for quivers without oriented cycles the category $\text{Rep}_K(Q)$ is hereditary. It turns out this result is true for general quiver.

**Theorem 2.4.** The category $\text{Rep}_K(Q)$ is hereditary, i.e. the subrepresentation of the projective representation is projective.

Theorem 2.4. follows from the following proposition

**Proposition 2.5.** The spaces $\text{Hom}_Q(V,W)$ and $\text{Ext}^1_Q(V,W)$ are the kernel (resp. cokernel) of the following map

$$d^V_W : \oplus_{x \in Q_0} \text{Hom}_K(V(x), W(x)) \to \oplus_{a \in Q_1} \text{Hom}_K(V(ta), W(ha)).$$

The map $d^V_W$ is given by the formula

$$d^V_W(\phi(x))_{x \in Q_0} = (\phi(ha)V(a) - W(a)\phi(ta))_{a \in Q_1}.$$

**Proof of Theorem 2.4** It follows from the proposition 2.5 and the snake lemma that the functor $\text{Ext}^1_Q(-, W)$ is right exact, which means that the $\text{Ext}^i_Q(V,W) = 0$ for $i \geq 2$. This, by standard results from homological algebra implies that the category is hereditary.

**Proof of Proposition 2.5.** The kernel of $d^V_W$ consists of families $\{\phi(x)\}_{x \in Q_0}$ such that for every $a \in Q_1$ we have $\phi(ha)V(a) - W(a)\phi(ta) = 0$. This, by definition is $\text{Hom}_Q(V,W)$. Let us look at the cokernel of the map $d^V_W$. It consists of the family of linear maps $\psi(a) : V(ta) \to W(ha)$ for every $a \in Q_1$. To each such family we associate the extension $E$ of $W$ by $V$. We define $E(x) := V(x) \oplus W(x)$ and the differential $E(a)$ is given in the block form by the matrix

$$\begin{pmatrix} V(a) & 0 \\ \psi(a) & W(a) \end{pmatrix}.$$ 

It is clear that the subspaces $\{W(x)\}_{x \in Q_0}$ form a subrepresentation of $E$ and that the factor is isomorphic to $V$. Thus we have an exact sequence

$$0 \to W \to E \to V \to 0.$$

Let us see when the family $\{\psi(a)\}$ defines a zero element in $\text{Coker}d^V_W$. This happens precisely when we have a family of linear maps $\{\phi(x)\}_{x \in Q_0}$ for which $\psi(a) = \phi(ha)V(a) - W(a)\phi(ta)$ for all $a \in Q_1$. We define the morphism

$$\eta : E \to V \oplus W$$

given over the vertex $x \in Q_0$ by the matrix

$$\eta(x) = \begin{pmatrix} 1_V & 0 \\ -\phi(x) & 1_W \end{pmatrix}.$$
It follows from the conditions on $\phi$ and $\psi$ that this is a morphism of representations. This morphism is obviously an isomorphism. We also have a commutative diagram of exact sequences

\[
0 \to W \to E \to V \to 0 \\
\downarrow 1_W \quad \downarrow \eta \quad \downarrow 1_V \\
0 \to W \to V \oplus W \to V \to 0
\]

This process can be reversed and every $\eta$ satisfying the commutative diagram above has to be of the triangular form written above, with the linear maps map $\phi(x)$ appearing in the lower left corner related to linear maps $\psi(a)$ by the relation $d^V_W([\{\phi(x)\}]) = \{\psi(a)\}$. Thus we identified $\text{Coker } d^V_W$ with $\text{Ext}^1_Q(V, W)$ in the Yoneda description.

**Corollary 2.6.** The Euler characteristic of $\chi(V, W)$ depends only on the dimension vectors of $V, W$ and it defines a nonsymmetric bilinear form (Euler form)

\[
\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).
\]

**Remark 2.7.** Notice that the Tits form is just the symmetrization of the Euler form.

In dealing with quivers that are not Dynkin or extended Dynkin one needs to use geometric methods, as the modules occur in more complicated families. One such approach comes from Invariant Theory.

We know that when $GL(Q, \beta)$ acts on $\text{Rep}(Q, \beta)$ the orbits of this action give the isomorphism classes of representations. However the scalar group $K^* \subset GL(Q, \beta)$ (consisting of identity matrices multiplied by the same nonzero scalar at each vertex) acts trivially on the set of isomorphism classes. This means that it is more convenient to consider the subgroup $SL(Q, \beta) := \prod_{x \in Q_0} SL(\beta(x), K)$ of $GL(Q, \beta)$.

The action of $GL(Q, \beta)$ on $\text{Rep}_K(Q, \beta)$ induces the action of $GL(Q, \beta)$ on the coordinate ring of polynomial functions on $\text{Rep}_K(Q, \beta)$ (which is just a polynomial ring generated by the entries of all matrices in a representation) by the rule

\[(g \circ f)(v) := f(g^{-1}v).
\]

The factor of $\text{Rep}(Q, \beta)$ by the action of $SL(Q, \beta)$ is described by the ring of $SL$-invariants

\[SI(Q, \beta) = \{ f \in K[\text{Rep}_K(Q, \beta)] \mid \forall g \in SL(Q, \beta) \quad g \circ f = f \}\]

which is isomorphic to the ring of $GL$- semi-invariants, i.e.

\[SI(Q, \beta) = \bigoplus_{\chi \in \text{char}GL(Q, \beta)} SI(Q, \beta)_\chi
\]
where
\[ SI(Q, \beta)_{\chi} = \{ f \in K[\text{Rep}_K(Q, \beta)] \mid \forall g \in GL(Q, \beta) \ g \circ f = \chi(g)f \} . \]

The characters of the group \( GL(Q, \beta) \) can be identified with the space \( \mathbb{Z}^{Q_0} \) because the character group of \( GL(n, K) \) is isomorphic to \( \mathbb{Z} \). We treat characters of \( GL(Q, \beta) \) as the functions on the dimension vectors. We refer to these characters as \textit{weights}.

Let us explain one more point of view explaining our interest in semi-invariants. Since the scalar group \( K^* \) acts trivially on isomorphism classes of representations it is natural to ask about the factor of the projectivisation of \( P(\text{Rep}_K(Q, \beta)) \) by the action of \( GL(Q, \beta) \). According to Geometric Invariant Theory the way to do it is to start with the \( GL(Q, \beta) \) linearized line bundle. These bundles correspond to weights \( \sigma \). Thus each weight \( \sigma \) allows to construct a Geometric Invariant Theory quotient which is just \( \text{Proj}(SI(Q, \beta, \sigma)) \) where

\[ SI(Q, \beta, \sigma) := \bigoplus_{n \geq 0} SI(Q, \beta)_{n\sigma}. \]

This means that the investigation of rings of semi-invariants gives information about the variation of this quotient with the weight \( \sigma \). In particular finding the weights where semi-invariants appear tells us for which \( \sigma \) the quotient is nonempty.

Before we start let us look at some examples.

**Example 2.8.** \( Q = A_1^{eq} \). Then easy calculation shows that if \( \beta(1) \neq \beta(2) \) then \( SI(Q, \beta) = K \). When \( \beta(1) = \beta(2) \) then \( SI(Q, \beta) = K[\text{det } V(a)] \), where \( \text{det } V(a) \) is a semi-invariant sending representation \( V \) to the determinant of the map \( V(a) \). The weight of \( \text{det } V(a) \) is \( (1, -1) \) because it corresponds to the representation \( \wedge^n V(1) \otimes \wedge^n V(2)^* \) inside of \( K[\text{Rep}_K(Q, \beta)] \). The weight \( \sigma \) is the first row for the Euler matrix

\[
E = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}
\]

i.e. it can be written in the form \( \langle \alpha, - \rangle \) for \( \alpha = (1, 0) \).

**Example 2.9.** More generally, let \( Q = A_n^{eq} \)

\[
Q: 1 \overset{a_1}{\rightarrow} 2 \overset{a_2}{\rightarrow} \ldots \overset{a_{n-1}}{\rightarrow} n
\]

The only possible semi-invariants are the semi-invariants \( V \mapsto \text{det}(V(a_{j-1} \circ \ldots \circ V(a_i))) \) of weights \( \sigma_{i,j} := \epsilon_i = \epsilon_j \) for \( 1 \leq i < j \leq n \). These are the weights \( \sigma_{i,j} = \langle \alpha_{i,j}, - \rangle \) where \( \alpha_{i,j} = \epsilon_i + \ldots + \epsilon_{j-1} \).

**Example 2.10.** Let \( Q = \xi(3) \). There are several possibilities for the semi-invariants: If \( \beta(1) = \beta(4) \) we have the semi-invariant \( V \mapsto \text{det } V(a) \) of weight \( (1, 0, 0, -1) \). There are three symmetric possibilities. If \( \beta(1) + \beta(2) = \beta(4) \) we have the semi-invariant \( V \mapsto \)
\[ \det (V(a), V(b)) \] of weight \((1, 1, 0, -1)\). There are three symmetric possibilities. If \(\beta(1) + \beta(2) + \beta(3) = \beta(4)\) we have the semi-invariant \(V \mapsto \det (V(a), V(b), V(c))\) of weight \((1, 1, 1, -1)\). Notice that the Euler matrix is

\[
E = \begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

All the weights of semi-invariants are the positive sums of rows of \(E\). This means these weights are of the form \(\langle \alpha, - \rangle\) for some dimension vectors. These dimension vectors turn out to be the dimensions of indecomposable representations for \(Q\).

Moreover, if we write the weight \(\sigma\) in the form \(\langle \alpha, - \rangle\) we see that the semi-invariants in the examples above occur only in weights for which \(\alpha\) is a dimension vector of an indecomposable representation.

This turns out to be the general pattern. In fact it is easy to produce the semi-invariants of such weights.

Let us recall that in Proposition 2.5, we defined the linear map

\[ d^V_W : \oplus_{x \in Q_0} \text{Hom}_K(V(x), W(x)) \to \oplus_{a \in Q_1} \text{Hom}_K(V(ta), W(ha)) \]

defined by the formula

\[ d^V_W(\phi(x))_{x \in Q_0} = (\phi(ha)V(a) - W(a)\phi(ta))_{a \in Q_1} \]

whose kernel and cokernel were \(\text{Hom}_Q(V, W)\) and \(\text{Ext}^1_Q(V, W)\) respectively. Let us denote \(\alpha = d(V)\), \(\beta = d(W)\). Let us assume that \(\langle \alpha, \beta \rangle = 0\). Then the linear map \(d^V_W\) is a square matrix. This allows to consider its determinant.

Let \(ux\) fix a representation \(V\). We define the function \(c^V : \text{Rep}_K(Q, \beta) \to K\) defined by the formula

\[ c^V(W) := \det(d^V_W). \]

Similarly, if we fix the representation \(W\) we can define the function \(c_W : \text{Rep}_K(Q, \alpha) \to K\) by the formula

\[ c_W(V) := \det(d^V_W). \]

**Proposition 2.11.**

a) The function \(c^V\) is a semi-invariant of weight \(\langle \alpha, - \rangle\), i.e. \(c^V \in SI(Q, \beta)_{\langle \alpha, - \rangle}\).

b) The function \(c_W\) is a semi-invariant of weight \(-\langle - , \beta \rangle\), i.e. \(c^V \in SI(Q, \alpha)_{-\langle - , \beta \rangle}\).

c) If

\[ 0 \to V' \to V \to V'' \to 0 \]
is an exact sequence of representations, with \( \alpha' := d(V') \), \( \alpha'' := d(V'') \), then if \( \langle \alpha', \beta \rangle = 0 \), then \( c^V = c'^V c''^V \). If \( \langle \alpha', \beta \rangle > 0 \), then \( c^V = 0 \).

d) If
\[
0 \rightarrow W' \rightarrow W \rightarrow W'' \rightarrow 0
\]
is an exact sequence of representations, with \( \beta' := d(V') \), \( \beta'' := d(V'') \), then if \( \langle \alpha, \beta' \rangle = 0 \), then \( c^W = c'^W c''^W \). If \( \langle \alpha, \beta' \rangle < 0 \), then \( c^W = 0 \).

Corollary 2.12.

a) If \( V = V' \oplus V'' \) then \( c^V \) is either a product \( c'^V c''^V \) or is identically zero,
b) If \( W = W' \oplus W'' \) then \( c_W \) is either a product \( c_W' c_W'' \) or is identically zero.

Corollary 2.13 (Reciprocity). Let \( \alpha \) and \( \beta \) be the dimension vectors satisfying \( \langle \alpha, \beta \rangle = 0 \). Then
\[
\dim SI(Q, \beta)_{\langle \alpha, - \rangle} = \dim SI(Q, \alpha)_{\langle - , \beta \rangle}.
\]

Proof. Let \( V_1, \ldots, V_s \) be the representations of dimension \( \alpha \) such that \( c^{V_1}, \ldots, c^{V_s} \) form a basis of \( SI(Q, \beta)_{\langle \alpha, - \rangle} \). These are linearly independent polynomial functions on \( Rep(Q, \beta) \), so there exist \( s \) representations \( W_1, \ldots, W_s \) from \( Rep(Q, \beta) \) such that \( det(c^{V_i}(W_j))_{1 \leq i, j \leq s} \) is nonzero. But \( c^{V_i}(W_j) = c_{W_j}(V_i) \). This proves that \( c_{W_1}, \ldots, c_{W_s} \) are linearly independent on \( Rep(Q, \alpha) \). This proves
\[
\dim SI(Q, \beta)_{\langle \alpha, - \rangle} \leq \dim SI(Q, \alpha)_{\langle - , \beta \rangle}.
\]
The other inequality is shown in the same way.●

The main result on semi-invariants is

Theorem 2.14. Let \( Q \) be a quiver with no oriented cycles.
a) For any dimension vector \( \gamma \) and a weight \( \sigma \) the weight space \( SI(Q, \gamma)_\sigma \) can be nonzero only for weights satisfying \( \sigma(\gamma) = 0 \),
b) If the weight \( \sigma \) is not of the form \( \sigma = \langle \alpha, - \rangle \), for some dimension vector \( \alpha \), then \( SI(Q, \beta)_\sigma = 0 \). If \( \sigma = \langle \alpha, - \rangle \) with \( \langle \alpha, \beta \rangle = 0 \), then \( SI(Q, \beta)_\sigma \) is spanned as a vector space by the semi-invariants \( c^V \) for \( V \in Rep_K(Q, \alpha) \),
c) If the weight \( \sigma \) is not of the form \( \sigma = -\langle - , \beta \rangle \), for some dimension vector \( \beta \), then \( SI(Q, \beta)_\sigma = 0 \). If \( \sigma = -\langle - , \beta \rangle \) with \( \langle \alpha, \beta \rangle = 0 \), then \( SI(Q, \alpha)_\sigma \) is spanned as a vector space by the semi-invariants \( c_W \) for \( W \in Rep_K(Q, \beta) \).

Before we prove the theorem we give some consequences. The first application is the description of the cone of weights in which the semi-invariants occur.

Let \( Q \) be a quiver with no oriented cycles and \( \beta \) - a dimension vector. We define the cone of weights
\[
\Sigma(Q, \beta) = \{ \sigma \in Char(GL(Q, \beta) \mid SI(Q, \beta)_\sigma \neq 0 \}.
\]
We also define the generic hom, ext and the generic subrepresentation. Let $Q$ be a quiver, $\alpha$, $\beta$ two dimension vectors. Notice that the sets

$$Z_0^t(\alpha, \beta) = \{(V, W) \in \text{Rep}(Q, \alpha) \times \text{Rep}(Q, \beta) \mid \dim \text{Hom}_Q(V, W) \geq t\}$$

$$Z_1^t(\alpha, \beta) = \{(V, W) \in \text{Rep}(Q, \alpha) \times \text{Rep}(Q, \beta) \mid \dim \text{Ext}_Q(V, W) \geq t\}$$

are closed, in fact they are defined by the determinantal condition for the matrix $d_{VW}$. Thus on the open set of $\text{Rep}(Q, \alpha) \times \text{Rep}(Q, \beta)$ the dimensions of $\text{Hom}_Q(V, W)$ and $\text{Ext}_Q(V, W)$ are constant. We define $\text{hom}_Q(\alpha, \beta)$ and $\text{ext}_Q(\alpha, \beta)$ to be the generic values of $\text{Hom}_Q(V, W)$ (resp. $\text{Ext}_Q(V, W)$) on $\text{Rep}(Q, \alpha) \times \text{Rep}(Q, \beta)$. We refer to them as generic hom and generic ext respectively.

We say that a dimension vector $\beta'$ is a sub-dimension vector of $\beta$ (notation: $\beta' \hookrightarrow \beta$) if every representation from $\text{Rep}(Q, \beta)$ has a subrepresentation of dimension vector $\beta'$. Similarly, we say that a dimension vector $\beta''$ is a factor-dimension vector of $\beta$ (notation: $\beta' \rightarrow \beta$) if every representation of dimension vector $\beta$ has a factor representation of dimension vector $\beta''$.

The connection between these notions is expressed by the following result.

**Theorem 2.15 (Schofield).** Let $\alpha, \beta$ be two dimension vectors for the quiver $Q$ without oriented cycles.

a) $\text{ext}_Q(\alpha, \beta) = 0$ if and only if $\alpha \hookrightarrow \alpha + \beta$,

b) $\text{ext}_Q(\alpha, \beta) \neq 0$ if and only if $\beta' \hookrightarrow \beta$ and $\langle \alpha, \beta - \beta' \rangle < 0$ for some dimension vector $\beta'$.

The main consequence of Theorem 2.14 is the explicit description of $\Sigma(Q, \beta)$ by means of one homogeneous linear equality and homogeneous linear inequalities.

**Theorem 2.16.** The set $\Sigma(Q, \beta)$ is a rational polyhedral cone described as follows

$$\Sigma(Q, \beta) = \{\sigma = \langle \alpha, - \rangle \mid \langle \alpha, \beta \rangle = 0, \forall \beta' \hookrightarrow \beta, \langle \alpha, \beta' \rangle \leq 0\}.$$ 

In particular $\Sigma(Q, \beta)$ is saturated, i.e. $n\sigma \in \Sigma(Q, \beta)$ for some $n \in \mathbb{Z}_+$ implies $\sigma \in \Sigma(Q, \beta)$.

**Remark 2.17.**

a) Theorems 2.14 implies that the ring $SI(Q, \beta)$ is generated (as an $K$-algebra) by the semi-invariants $c^V$ where $V$ is an indecomposable representation of $Q$ with $\langle d(V), \beta \rangle = 0$.

b) In Theorem 2.16. it is enough to consider the subrepresentations $\beta' \hookrightarrow \beta$ for which general representation of dimension $\beta'$ is indecomposable. With this restriction still a lot of inequalities $\sigma(\beta') \leq 0$ are redundant.
Now we apply Theorem 2.16 to the problem 3). Take the quiver $Q$ to be the following quiver

$\xymatrix{ x_1 \ar[r]^{a_1} & x_2 \ar[r]^{a_2} & \cdots & x_{n-1} \ar[r]^{a_{n-1}} & u \ar[d]^{c_{n-1}} \ar[l]^{b_{n-1}} \ar[d]^{z_{n-1}} & y_{n-1} \ar[l]^{b_2} \ar[d]^{c_{n-2}} & \cdots & y_2 \ar[l]^{b_1} \ar[d]^{c_1} & z_2 \ar[d]^{c_1} & z_1 }$

and the dimension vector $\beta(x_i) = \beta(y_i) = \beta(z_i) = i$, $\beta(u) = n$. A standard calculation from representation theory shows that the ring $SI(Q, \beta)$ can be described as

$$SI(Q, \beta) = \oplus_{\lambda, \mu, \nu} (V(\lambda)^* \otimes V(\mu)^* \otimes V(\nu))^{SL(n, \mathbb{C})},$$

with the space $(V(\lambda)^* \otimes V(\mu)^* \otimes V(\nu))^{SL(n, \mathbb{C})}$ occurring in the weight $\sigma$ with $\sigma(x_i) = \lambda_i - \lambda_{i-1}$, $\sigma(y_i) = \mu_i - \mu_{i-1}$, $\sigma(z_i) = -\nu_i + \nu_{i-1}$, $\sigma(u) = 0$. Here $SL(n, \mathbb{C})$ is the special linear group at the central vertex $u$. Notice that the space of invariants $(V(\lambda)^* \otimes V(\mu)^* \otimes V(\nu))^{SL(n, \mathbb{C})}$ is non-zero if and only if $V(\nu)$ occurs in the tensor product $V(\lambda) \otimes V(\mu)$. Thus Theorem 2.16 implies that the set of triples $(\lambda, \mu, \nu)$ such that $V(\nu)$ occurs in the tensor product $V(\lambda) \otimes V(\mu)$ is saturated.

### 3. The geometry of orbit closures for the Dynkin quivers.

We are interested in orbit closures of representations of Dynkin quivers. The archetype case are the orbit closures for the orbits of the quiver $A_1^{eq}$.

**Example 3.1.** $Q : 1 \to 2$. For a fixed dimension vector $\beta = (n, m)$ the space $Rep_K(Q, \beta)$ is just a set of $m \times n$ matrices where the product of two general linear groups acts by row and column operations. The orbits are the matrices of fixed rank. This means that the orbit closures are just determinantal varieties

$$X_r = \{ \phi : K^n \to K^m \mid \text{rank}(\phi) \leq r \}.$$
The determinantal varieties have many nice properties which are interesting for commutative algebraists. In particular:

- They are normal. This means the coordinate rings $K[X_r]$ are integrally closed in their fields of fractions.
- They are Cohen-Macaulay. This means by cutting by hypersurfaces we can reduce the study of $X_r$ to the study of zero dimensional schemes (i.e. the coordinate ring $K[X_r]$ divided by well chosen regular sequence is finite dimensional over $K$).
- They have rational singularities. This means that on the desingularization of $X_r$ the higher cohomology of the structure sheaf vanishes. This property implies Cohen-Macaulay. In particular the varieties $X_r$ have two nice desingularizations
  
  $$Z_r = \{(\phi, R) \in \text{Hom}_K(K^n, K^m) \times \text{Grass}(n-r, K^n) \mid R \subset \text{Ker}(\phi)\},$$
  $$Z'_r = \{(\phi, S) \in \text{Hom}_K(K^n, K^m) \times \text{Grass}(r, K^m) \mid S \supset \text{Im}(\phi)\}$$

  for which one can prove rational singularities rather easily.
- The determinantal varieties are Gorenstein (meaning Cohen-Macaulay plus selfduality of free resolution of $K[X_r]$ as a module over the coordinate ring of $\text{Rep}_K(Q, \beta)$) if $r = 0$ or if $r \geq 1$ and $m = n$.

In recent years there have been important progress made in the direction of generalizing these properties to orbit closures of representations of Dynkin quivers.

There are even more basic algebraic questions: Let me list some of them:

1) When is one orbit contained in the closure of another orbit ?

2) What are the defining ideals of the orbit closures (in the case of determinantal variety this is the ideal of $(r+1) \times (r+1)$ minors of a generic $m \times n$ matrix).

Let us employ the following notation. For a representation $V$ of dimension vector $\beta$ we denote by $O_V$ its orbit in $\text{Rep}_K(Q, \beta)$. We also denote $\text{Ind}(Q)$ the set of isomorphism classes of indecomposable representations of $Q$.

Question 1) was settled by Bongartz.

**Theorem 3.2.** Let $V, W$ be two representations of dimension vector $\beta$ of a Dynkin quiver $Q$. Then

$$O_V \subset O_W \iff \forall U \in \text{Ind}(Q) \ dim(\text{Hom}_Q(U, V)) \geq dim(\text{Hom}_Q(U, W)).$$

This means we can give set-theoretic conditions to be in the orbit closure $O_V$. Let $U_\alpha$ be an indecomposable representation corresponding to the positive root $\alpha$. Let $r_\alpha = dim(\text{Hom}_Q(U_\alpha, V))$. Then

$$\bar{O}_V = \{W \mid \forall \alpha \ dim(\text{Hom}_Q(U_\alpha, W)) \geq r_\alpha\}.$$

This means set-theoretically the orbit $O_V$ is given by the vanishing of $r_\alpha + 1$ size minors of the matrix $d_{ij}^{\mu\nu}$.  

20
Example 3.3. Let $Q = A_{eq}^n$. Then the rank conditions are just conditions for the rank $r_{i,j}$ of the compositions $V(i) \to V(j)$ for each $1 \leq i \leq j \leq n$. Here $r_{i,i} := \text{dim}(V(i))$. We can determine the representation $V$ by multiplicities of the indecomposable summands, $V = \oplus m_{i,j} E_{i,j}$ or by the ranks $r_{i,j}$ of the maps $V(i) \to V(j)$. These two types of conditions easily translate to each other. We have

$$r_{i,j} = \sum_{k \leq i \leq j \leq l} m_{k,l}, m_{i,j} = r_{i,j} - r_{i,j+1} - r_{i-1,j} + r_{i-1,j+1}.$$ 

The last equation tells us the necessary conditions for the set of ranks $r_{i,j}$ actually occurring. The condition in terms of $m_{i,j}$’s is obviously $m_{i,j} \geq 0$.

The orbit closures of Dynkin quivers contain many interesting varieties. For example the Schubert varieties in partial flag varieties are among them.

Example 3.4. Let us consider the partial flag variety $\text{Flag}(r_1, r_2, \ldots, r_s; K^n)$ of flags

$$0 = F_0 \subset F_1 \subset \ldots \subset F_s \subset K^n$$

with $\text{dim}(F_i) = r_i$. The Schubert varieties are the orbits of the group $B$ of upper-triangular matrices in $\text{GL}(n, C)$ acting on $\text{Flag}(r_1, \ldots, r_s; K^n)$ (we identify $\text{Flag}(r_1, \ldots, r_s; K^n)$ with $\text{GL}(n, C)/P(r_1, \ldots, r_s)$). We take a quiver

$$Q : x_1 \to x_2 \to \ldots x_{n-1} \to u \leftarrow y_s \leftarrow \ldots \leftarrow y_2 \leftarrow y_1$$

and the dimension vectors $\beta(x_i) = i, \beta(u) = n, \beta(y_j) = r_j$. Then the intersection of the orbit closure with the open set of representations with all linear maps being injective gives the fibered product $Y \times_B \text{GL}(n, C)$ for some Schubert variety $Y$. All Schubert varieties can be obtained in this way. Thus the study of the singularities of Schubert varieties of type $A_n$ is part of the study of the singularities of orbit closures for quivers of type $A_n$.

Remark 3.5. For some time in the seventies there was hope that among different Dynkin quivers one would find the singularities of types $A_n, D_n, E_6, 7, 8$ as defined in singularities theory. This did not materialize, Bongartz showed that the minimal degenerations for all quivers of Dynkin type are the same ($m \times n$ matrices of rank $\leq 1$ at the origin).

Here are the results known for orbit closures of Dynkin quivers right now.

Theorem 3.6. (Abbeassis, Del Fra, Kraft [ADK], 1982, characteristic zero case; Lakshmibai, Magyar [LM], 1999, general case) If $Q = A_{eq}^n$ then all orbit closures are Cohen-Macaulay and the rank conditions generate the defining ideals. If $\text{Char}(K) = 0$ all these orbit closures have rational singularities.

The method of Lakshmibai and Magyar is to exhibit all orbit closures as open subsets of Schubert varieties of type $A_n$ and use the conclusions of standard monomial theory.
Theorem 3.7. (Bobiński, Zwara). If $Q$ is a quiver of type $A_n$ then all orbit closures are Cohen-Macaulay and the rank conditions generate the defining ideals. If $\text{Char}(K) = 0$ all these orbit closures have rational singularities.

This result is based on the general result of Zwara on the representations of Artin algebras.

Theorem 3.8 (Zwara). Let $\phi : A \to B$ be a homomorphism of Artin algebras. Let $\tilde{\phi} : B - \text{mod} \to A - \text{mod}$ be the natural functor of the restriction of scalars. Assume that $\tilde{\phi}$ is

a) exact,

b) $\text{Hom}$-controlled, i.e. there exists a bilinear form $\xi : K_0(A) \times K_0(A) \to \mathbb{Z}$ such that

$$\dim(\text{Hom}_A(\tilde{\phi}(V), \tilde{\phi}(W)) - \dim(\text{Hom}_B(V, W)) = \xi([V], [W]).$$

Then the singularities of $\mathcal{O}_V$ and of $\mathcal{O}_{\tilde{\phi}(V)}$ are smoothly equivalent. In particular Cohen-Macaulay, rational singularities properties are preserved.

In the case $B = KQ$, $A = KQ'$ Bobiński and Zwara construct needed functor $\phi$ for all embeddings of $AR$ quiver of $Q$ into the $AR$ quiver of $Q'$. Thus to get the orbit closures for quiver of type $A_n$ with strange orientation, we imbed its $AR$-quiver into the $AR$ quiver of bigger equioriented $A_n$ quiver.

Example 3.9.