1. Prove that $U(5)$ is not isomorphic to $U(8)$.

- $U(5) = \{i \mid 1 \leq i \leq 5, \gcd(i, 5) = 1\} = \{1, 2, 3, 4\}$, operation is multiplication (mod 5).
  - Subgroup generated by 2: $<2> = \{2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16 = 1\}$
  - Therefore $U(5)$ is cyclic since it can be generated by one single element.
- $U(8) = \{i \mid 1 \leq i \leq 8, \gcd(i, 8) = 1\} = \{1, 3, 5, 7\}$, operation is multiplication (mod 8).
  - $<1> = \{1^1 = 1\} = \{1\} = U(8)$
  - $<3> = \{3^1 = 3, 3^2 = 9 = 1\} = \{3, 1\} = U(8)$
  - $<5> = \{5^1 = 5, 5^2 = 25 = 1\} = \{5, 1\} = U(8)$
  - $<7> = \{7^1 = 7, 7^2 = 49 = 1\} = \{7, 1\} = U(8)$
  - Therefore $U(8)$ is not cyclic since it can not be generated by one single element.
- Thm. Suppose $G \cong G'$ are isomorphic groups. Then $G$ is cyclic if and only if $G'$ is cyclic.
- Since $U(5)$ is cyclic and $U(8)$ is not cyclic they cannot be isomorphic groups.

2. Find all automorphisms of $\mathbb{Z}_{10}$. Make sure that you explain your work.

- $(\mathbb{Z}_{10}, +)$ is a cyclic group with a generator 1.
- Any automorphism of a cyclic group must send a generator to a generator.
- All generators of $(\mathbb{Z}_{10}, +)$ are $\{i \mid 1 \leq i \leq 10, \gcd(i, 10) = 1\} = \{1, 3, 7, 9\}$
- All different automorphisms of $(\mathbb{Z}_{10}, +)$ are:
  - $f_1 : (\mathbb{Z}_{10}, +) \to (\mathbb{Z}_{10}, +)$ such that $f_1(1) = 1$
    $f_1(x) = 1x$
  - $f_2 : (\mathbb{Z}_{10}, +) \to (\mathbb{Z}_{10}, +)$ such that $f_2(1) = 3$
    $f_2(x) = 3x$
  - $f_3 : (\mathbb{Z}_{10}, +) \to (\mathbb{Z}_{10}, +)$ such that $f_3(1) = 7$
    $f_3(x) = 7x$
  - $f_4 : (\mathbb{Z}_{10}, +) \to (\mathbb{Z}_{10}, +)$ such that $f_4(1) = 9$
    $f_4(x) = 9x$
3. Let $G = D_4 = \langle \rho, t \mid \rho^4 = e, t^2 = e, tpt = \rho^{-1} \rangle$ be the dihedral group with the distinct elements: $\{e, \rho, \rho^2, \rho^3, t, t\rho, t\rho^2, t\rho^3\}$.
Let $H = \langle t\rho \rangle$ be the subgroup generated by $t\rho$.

(a) What are the elements of $H$? (Explain!)
- $\rho^4 = e$ implies $\rho^{-1} = \rho^3$ and $t^2 = e$ implies $t^{-1} = t$
- $(tpt = \rho^{-1} \text{ and } \rho^{-1} = \rho^3) \Rightarrow (tpt = \rho^3) \Rightarrow (t = t\rho^3)$
- $(t\rho)^1 = t\rho, (t\rho)^2 = t\rho t\rho = t(\rho t)\rho = t(t\rho^3)\rho = t^2\rho^4 = ee = e$
- $H = \langle t\rho \rangle = \{t\rho, (t\rho)^2 = e\}$

(b) What is the size of each of the left cosets of $H$ in $G$. (Explain!)
- $|aH| = |H| = 2$

(c) Find two distinct left cosets of $H$ in $G$.
- $eH = \{et\rho, ee\} = \{t\rho, e\} = H$
- $tH = \{tt\rho, te\} = \{\rho, t\} \neq eH$

4. Let $G = S_4$ be the permutation group on four elements $\{1, 2, 3, 4\}$.
Let $H = \langle (1342) \rangle$ be the subgroup generated by the permutation $\alpha = (1342)$.

(a) What is the size of $H$? (Explain!)
- (Method 1)
  - $(1342)$ is a 4-cycle, therefore it has order 4, i.e. $|(1342)|$
  - $|H| = |\langle (1342) \rangle| = 4$ (use the fact that $|a| = |\langle a \rangle|$)
- (Method 2)
  - Or, we can compute:
    $$(1342)^1 = (1342), (1342)^2 = (14)(32), (1342)^3 = (1243), (1342)^4 = (1)$$
  - $H = \{(1342), (14)(32), (1243), (1)\}$ and therefore $|H| = 4$

(b) How many right cosets of $H$ in $G$ are there? (Explain!)
- (# of right cosets) $= [G : H] = |G|/|H| = 4!/4 = 24/4 = 6$

(c) Find two right cosets of $H$ in $G$.
- $H(1) = \{(1342)(1), (14)(32)(1), (1243)(1), (1)\} = \{(1342), (14)(32), (1243), (1)\} = H$
- $H(12) = \{(1342)(12), (14)(32)(12), (1243)(12), (12)\}$
  $H(12) = \{(234), (1324), (143), (12)\} \neq H$

There are many more questions and answers about the same two groups $D_4, S_4$ and their cosets in the Solutions to Practice Quiz.