**Dihedral group $D_4$**

1. Let $D_4 = \langle \rho, t \mid \rho^4 = e, t^2 = e, tpt = \rho^{-1} \rangle$ be the dihedral group.

   (a) Write the Cayley table for $D_4$. You may use the fact that $\{e, \rho, \rho^2, \rho^3, t, \rho t, \rho^2 t, \rho^3 t\}$ are all distinct elements of $D_4$.

   **Table 1: $D_4$**

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   (b) Find the subgroup generated by $\{t^2\}$. **Done in class.**

   (c) Find the subgroup generated by $\{t, t^2\}$.

   First notice that the subgroup generated by $\{t, t^2\}$, must contain $t^2 = e$ and also $tt^2 = \rho^2$.

   Next, make the Cayley table of the elements $\{e, t, t^2, \rho^2\}$ and check if it is closed under operation and inverses.

   **Table 2: Subgroup $\langle t, t^2 \rangle$ of $D_4$**

   | $\langle t, t^2 \rangle$ | $e$ | $t$ | $t^2$ | $\rho^2$ |
   |---|---|---|---|
   | $e$ | $e$ | $t$ | $t^2$ | $\rho^2$ |
   | $t$ | $t$ | $\rho^2$ | $t^2$ | |
   | $t^2$ | $t^2$ | $\rho^2$ | $e$ | $t$ |
   | $\rho^2$ | $\rho^2$ | $t^2$ | $t$ |

   Therefore $S = \{e, t, t^2, \rho^2\}$ is a nonempty subset of $D_4$ closed under operation and inverses. So $S$ is a subgroup of $D_4$ by one of the subgroup theorems ("two step subgroup theorem").

   (d) Find the order of each of the elements of $D_4$.

   From the Table for $D_4$ it follows that the orders of the elements of $D_4$ are as follows (i.e. these are the smallest positive integers such that $a^n = e$):

   $|e| = 1$, $|\rho| = 4$, $|\rho^2| = 2$, $|\rho^3| = 4$, $|t| = 2$, $|t\rho| = 2$, $|t\rho^2| = 2$, $|t\rho^3| = 2$. 


Isomorphisms

2. Prove that the map $f : (\mathbb{Z}_{10}, +) \to (\mathbb{Z}_{10}, +)$ defined by $f(x) = x + 2 \pmod{10}$ is not an isomorphism of groups.

- One way: $f(x)$ is NOT group isomorphism, since
  $f(x + y) = 2 + x + y$, but $f(x) + f(y) = 2 + x + 2 + y$, and
  $2 + x + y \neq 4 + x + y = 2 + x + 2 + y$ in $\mathbb{Z}_{10}$.
  Therefore the property $f(ab) = f(a)f(b)$ does not hold.

- Another way: For each isomorphism of groups $f : G \to G'$, $f(e_G) = e_{G'}$. However $f(0) = 2 \neq 0$, hence $f$ is not isomorphism.

3. Prove that the map $f : (\mathbb{Z}_{10}, +) \to (\mathbb{Z}_{10}, +)$ defined by $f(x) = 3x \pmod{10}$ is an isomorphism of groups.

- Isomorphisms between cyclic groups $G = \langle a \rangle$ and $G' = \langle b \rangle$ of the same order can be defined by
  - sending $a$, the generator of group $G$ to a generator of $G'$ and
  - defining $f(a^i) := (f(a))^i$.
- $(\mathbb{Z}_{10}, +)$ is a cyclic group of order 10 with a generator 1.
- Generators of $(\mathbb{Z}_{10}, +)$ are all integers $\{k \mid 1 \leq k \leq 9, \gcd(k, 10) = 1\}$
- $f(1) = 3$ which is also a generator in $(\mathbb{Z}_{10}, +)$
- $f(a^i) = (f(a))^i$ in additive notation with generator 1, is written as
  $f(i \cdot 1) = i \cdot f(1)$, which is the same as $f(x) = x \cdot 3$ or $f(x) = 3x$.
- Therefore $f$ is an isomorphism of the above cyclic groups.

4. Let $G = \langle a \rangle$ be a cyclic group of order 10. Prove that the map $f : G \to G$ defined by $f(a) = a^3$ and $f(a^i) = a^{3i}$ is a group isomorphism.

- Isomorphisms between cyclic groups $G = \langle a \rangle$ and $G' = \langle b \rangle$ of the same order can be defined by
  - sending $a$, the generator of group $G$ to a generator of $G'$ and
  - defining $f(a^i) := (f(a))^i$.
- $G = \langle a \rangle$ is a cyclic group of order 10 with a generator $a$.
- Generators of $\langle a \rangle$ are all $\{a^t \mid 1 \leq t \leq 9, \gcd(t, 10) = 1\}$
- $f(a) = a^3$ which is also a generator of $G = \langle a \rangle$
- From $f(a^i) = a^{3i}$ we have $f(a^i) = a^{3i} = (a^3)^i = (f(a))^i$
- Therefore $f$ is an isomorphism of the above cyclic groups.
5. Let $G = \langle a \rangle$ be a cyclic group of order 10. Prove that the map $f : G \to G$ defined by $f(a) = a^4$ and $f(a^i) = a^{4i}$ is not group isomorphism.

- **(One way)** Isomorphism must send generator to a generator (see previous problems) but $a^4$ is not generator the cyclic group of order 10, $G = \langle a \rangle$ since $\gcd(4, 10) = 2 \neq 1$.
- **(Another way)** Orders of $a$ and $f(a)$ must be the same if $f$ is an isomorphism. But $|a| = 10$ and $|f(a)| = |a^{\gcd(4,10)}| = |a^2| = \frac{10}{2} = 5$. Hence $|a| \neq |f(a)|$. Hence $f$ is not an isomorphism.
- **(Third way)** The map $f$ is not one-to-one.
  
  Proof: $f(a) = a^4$ and $f(a^7) = a^{14} = a^4$ but $a \neq a^7$ in a cyclic group of order 10 (all elements $\{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9\}$ are distinct).

6. Let $G = \langle a \rangle$ be a cyclic group of order 10. Describe all automorphisms of $G$.

- Hint: $f(a)$ must be a generator of $G = \langle a \rangle$.

7. Describe 3 different group isomorphisms $(\mathbb{Z}_{10}, +) \to (\mathbb{Z}_{10}, +)$. *(Make sure that you explain why they are isomorphisms!)*

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8. Are the groups $S_3$ and $U(9)$ isomorphic? Prove your statement.

- No.
  
  - Theorem: Let $G \cong G'$ be isomorphic groups. Then $G$ is abelian if and only if $G'$ is abelian.
  
  - $S_3$ is non-abelian and $U(9)$ is abelian.

9. Are the groups $S_3$ and $D_3$ isomorphic? Prove your statement.

- Make a table for $D_3$
- Make a table for $S_3$
- Define a map: $f(\rho) = (123)$, $f(\rho^2) = (132)$, $f(\rho^3) = (1)$, $f(t) = (12)$, $f(t\rho) = (12)(123) = (23)$, $f(t\rho^2) = (12)(132) = (13)$
- Check that the Cayley multiplication tables are the same under this identification (function).

10. Are the groups $U(5)$ and $U(10)$ isomorphic? Prove your statement.

- $U(5) = \{1, 2, 3, 4\}$, integers relatively prime to 5 and between 1 and 4.
- $2^1 = 2, 2^2 = 4, 2^3 = 3, 2^4 = 1$ implies that 2 generates all of $U(5)$
- $U(5)$ is cyclic group of order 4, therefore isomorphic to $(\mathbb{Z}_4, +)$.
- $U(10) = \{1, 3, 7, 9\}$, integers relatively prime to 10 and between 1 and 9.
• $3^1 = 3, \ 3^2 = 9, 3^3 = 7, \ 3^4 = 1$ implies that 3 generates all of $U(10)$
• $U(10)$ is cyclic group of order 4, therefore isomorphic to $(\mathbb{Z}_4, +)$. 
• Both $U(5)$ and $U(10)$ are isomorphic to $(\mathbb{Z}_4, +)$, therefore they are isomorphic.
11. Let $G = D_4$ Let $D_4 = \langle \rho, t \mid \rho^4 = e, t^2 = e, tpt = \rho^{-1} \rangle$ be the dihedral group with the distinct elements: \{e, \rho, \rho^2, \rho^3, t, \rho t, t\rho^2, t\rho^3\}. Let $H = \langle t \rangle$ be the subgroup generated by $t$.

(a) What are the elements of $H$?
- $H = \{t, t^2 = e\} = \{e, t\}$

(b) What is the size of $H$?
- $|H| = |\{e, t\}| = 2$

(c) What is the size of each of the left cosets of $H$ in $G$.
- $|aH| = |H| = 2$

(d) Find all left cosets of $H$ in $G$.
- $eH = \{e, t\} = H$
- $tH = \{te, tt\} = \{t, e\} = H$
- $\rho H = \{\rho e, \rho t\} = \{\rho, t\rho^3\}$ (use $\rho t = t\rho^3$ for this and other cosets)
- $\rho^2 H = \{\rho^2 e, \rho^2 t\} = \{\rho^2, t\rho^2\}$
- $\rho^3 H = \{\rho^3 e, \rho^3 t\} = \{\rho^3, t\rho\}$
- $t\rho H = \{t\rho e, t\rho t\} = \{t\rho, \rho^3\}$
- $t\rho^2 H = \{t\rho^2 e, t\rho^2 t\} = \{t\rho^2, \rho^2\}$
- $t\rho^3 H = \{t\rho^3 e, t\rho^3 t\} = \{t\rho^3, \rho\}$
- Notice: $eH = tH = H$, $\rho H = t\rho^3 H$, $\rho^2 H = t\rho^2 H$, $\rho^3 H = t\rho H$

(e) How many left cosets of $H$ in $G$ are there?
- $\# \text{ of (distinct) cosets of } H \text{ in } G = [G : H] = |G|/|H| = 8/2 = 4$
- Also notice that we computed above 4 distinct cosets.

(f) What is the size of each of the right cosets of $H$ in $G$.
- $|Ha| = |H| = 2$

(g) Find all right cosets of $H$ in $G$.
- $He = \{e, t\} = H$
- $Ht = \{et, tt\} = \{t, e\} = H$
- $H\rho = \{e\rho, t\rho\} = \{\rho, t\rho\}$
- $H\rho^2 = \{e\rho^2, t\rho^2\} = \{\rho^2, t\rho^2\}$
- $H\rho^3 = \{e\rho^3, t\rho^3\} = \{\rho^3, t\rho^3\}$
- $Ht\rho = \{et\rho, t\rho t\} = \{t\rho, \rho\}$
- $Ht\rho^2 = \{et\rho^2, t\rho^2 t\} = \{t\rho^2, \rho^2\}$
- $Ht\rho^3 = \{et\rho^3, t\rho^3 t\} = \{t\rho^3, \rho^3\}$
• Notice: \( He = Ht = H, \quad H\rho = Ht\rho, \quad H\rho^2 = Ht\rho^2, \quad H\rho^3 = Ht\rho^3 \)

(h) How many right cosets of \( H \) in \( G \) are there?

• The number of (distinct) right cosets of \( H \) in \( G \) is \( [G : H] = |G|/|H| = 8/2 = 4 \)

• Also notice that we computed above 4 distinct right cosets.

(i) Are the left cosets of \( H \) in \( G \) the same as the right cosets of \( H \) in \( G \)?

• Some are the same but some are not:
  
  • \( eH = He, \quad tH = Ht, \quad \rho^2H = H\rho^2, \quad t\rho^2H = Ht\rho^2 \).
  
  • \( \rho H \neq H\rho, \quad \rho^3 H \neq H\rho^3, \quad t\rho H \neq Ht\rho, \quad t\rho^3 H \neq Ht\rho^3 \).
12. Let $G = \mathbb{Z}_{12}$ be the group under addition ($mod 12$). Let $H = \langle 3 \rangle$ be the subgroup generated by 3.

(a) What are the elements of $H$?
   - $H = \{0, 3, 6, 9\}$

(b) What is the size of $H$?
   - $|H| = |\{0, 3, 6, 9\}| = 4$

(c) What is the size of each of the left cosets of $H$ in $G$.
   - $|a + H| = |H| = 4$ (Notice additive notation!)

(d) Find all left cosets of $H$ in $G$.
   - $0 + H = \{0, 3, 6, 9\}$
   - $1 + H = \{1, 4, 7, 10\}$
   - $2 + H = \{2, 5, 8, 11\}$
   - $3 + H = \{3, 6, 9, 0\} = 0 + H = 6 + H = 9 + H$
   - $4 + H = \{4, 7, 10, 1\} = 1 + H = 7 + H = 10 + H$
   - $5 + H = \{5, 8, 11, 2\} = 2 + H = 8 + H = 11 + H$

(e) How many left cosets of $H$ in $G$ are there?
   - # of (distinct) left cosets of $H$ in $G = [G : H] = |G|/|H| = 12/4 = 3$
   - Also notice that we computed above 3 distinct left cosets.

(f) What is the size of each of the right cosets of $H$ in $G$.
   - $|H + a| = |H| = 4$ (Notice additive notation!)

(g) Find all right cosets of $H$ in $G$.
   - $H + 0 = \{0, 3, 6, 9\} = H + 3 = H + 6 = H + 9$
   - $H + 1 = \{1, 4, 7, 10\} = H + 4 = H + 7 = H + 10$
   - $H + 2 = \{2, 5, 8, 11\} = H + 5 = H + 8 = H + 11$

(h) How many right cosets of $H$ in $G$ are there?
   - # of (distinct) right cosets of $H$ in $G = [G : H] = |G|/|H| = 12/4 = 3$
   - Also notice that we computed above 3 distinct right cosets.

(i) Are the left cosets of $H$ in $G$ the same as the right cosets of $H$ in $G$?
   - YES.
   - $0 + H = H + 0, \ 1 + H = H + 1, \ 2 + H = H + 2$, and also $a + H = H + a$ for all $a \in G = \mathbb{Z}_{12}$. 


13. Let $G = S_4$ be the permutation group on four elements \{1, 2, 3, 4\}. Let $H = \langle (1342) \rangle$ be the subgroup generated by the permutation $\alpha = (1342)$.

(a) What are the elements of $H$?
   \[ H = \{(1342), (14)(32), (1243), (1)\} \]

(b) What is the size of $H$?
   \[ |H| = |\{(1342), (14)(32), (1243), (1)\}| = 4 \]

(c) What is the size of each of the left cosets of $H$ in $G$.
   \[ |aH| = |H| = 4 \]

(d) Find all left cosets of $H$ in $G$.
   \[ (1)H = \{(1342), (14)(32), (1243), (1)\} = (1342)H = (14)(32)H = (1243)H = H \]
   \[ (12)H = \{(12)(1342), (12)(14)(32), (12)(1243), (12)(1)\} = \{(134), (1423), (243), (12)\} \]
   etc.

(e) How many distinct left cosets of $H$ in $G$ are there?
   \[ \# \text{ of (distinct) left cosets of } H \text{ in } G = [G : H] = |G|/|H| = 24/4 = 6 \]

(f) What is the size of each of the right cosets of $H$ in $G$.
   \[ |Ha| = |H| = 4 \]

(g) Find all right cosets of $H$ in $G$.

(h) How many right cosets of $H$ in $G$ are there?

(i) Are the left cosets of $H$ in $G$ the same as the right cosets of of $H$ in $G$?
14. Let $G = U(12)$ be the group of invertible integers ($mod\ 12$).
Let $H = \langle 5 \rangle$ be the subgroup generated by the element (5).

(a) What are the elements of $H$?

(b) What is the size of $H$?

(c) What is the size of each of the left cosets of $H$ in $G$.

(d) Find all left cosets of $H$ in $G$.

(e) How many left cosets of $H$ in $G$ are there?

(f) What is the size of each of the right cosets of $H$ in $G$.

(g) Find all right cosets of $H$ in $G$.

(h) How many right cosets of $H$ in $G$ are there?

(i) Are the left cosets of $H$ in $G$ the same as the right cosets of $H$ in $G$?
15. Let $G = A_4$ be the group of even permutations on four elements $\{1, 2, 3, 4\}$. Let $H = \langle (134) \rangle$ be the subgroup generated by the permutation $\alpha = (134)$.

(a) What are the elements of $H$?

(b) What is the size of $H$?

(c) What is the size of each of the left cosets of $H$ in $G$.

(d) Find all left cosets of $H$ in $G$.

(e) How many left cosets of $H$ in $G$ are there?

(f) What is the size of each of the right cosets of $H$ in $G$.

(g) Find all right cosets of $H$ in $G$.

(h) How many right cosets of $H$ in $G$ are there?

(i) Are the left cosets of $H$ in $G$ the same as the right cosets of $H$ in $G$?
16. Let $G$ be the group generated by the permutations $\alpha = (1\ 2\ 3\ 4)$ and $\beta = (5\ 6\ 7\ 8\ 9\ 10)$ in $S_{10}$. Let $H = \langle \alpha\beta \rangle$ be the subgroup of $G$ generated by the permutation $\alpha\beta = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8\ 9\ 10)$.

(a) What are the elements of $H$?

(b) What is the size of $H$?

(c) What is the size of each of the left cosets of $H$ in $G$.

(d) Find all left cosets of $H$ in $G$.

(e) How many left cosets of $H$ in $G$ are there?

(f) What is the size of each of the right cosets of $H$ in $G$.

(g) Find all right cosets of $H$ in $G$.

(h) How many right cosets of $H$ in $G$ are there?

(i) Are the left cosets of $H$ in $G$ the same as the right cosets of $H$ in $G$?