1. Let $S_n$ be the group of permutations on $n$ elements $\{1, 2, 3, \ldots, n\}$. Let $A_n$ be the subgroup of even permutations. Prove that $A_n$ is normal subgroup of $S_n$.

2. Let $S_{15}$ be the group of permutations on 15 elements $\{1, 2, 3, \ldots, 15\}$. Prove that $S_{15}$ has elements of order 56 but does not have any elements of orders 49 or 50.

Answer: Use the following facts:

(a) Order of a $k$-cycle is $k$, i.e. $|\alpha| = k$ for $\alpha$ a $k$-cycle.
(b) $|\alpha \beta| = \text{lcm}(|\alpha|, |\beta|)$ if $\alpha$ and $\beta$ are disjoint permutations.

- $S_{15}$ has elements of order 56: Let $\sigma = \alpha \beta$, where $\alpha$ is an 8-cycle and $\beta$ is a 7-cycle. Then $|\sigma| = |\alpha \beta| = \text{lcm}(|\alpha|, |\beta|) = \text{lcm}(8, 7) = 56$.

- $S_{15}$ does not have any elements of orders 49: In order for $\text{lcm}(m, n) = 49$, either $m = 49$ or $n = 49$. Therefore we would need a cycle of order 49, hence need a 49-cycle in $S_{15}$, which is impossible.

- $S_{15}$ does not have any elements of orders 50: In order for $\text{lcm}(m, n) = 50$, either $m$ or $n$ must be a multiple of $5^2 = 25$. Therefore we would need a 25-cycle or 50-cycle in $S_{15}$, which is impossible.

3. Prove that a factor group of a cyclic group is cyclic.

Answer: Recall: A group $G$ is cyclic if it can be generated by one element, i.e. if there exists an element $a \in G$ such that $G = \langle a \rangle$ (this means that all elements of $G$ are of the form $a^i$ for some integer $i$.)

Recall: Elements of a factor group $G/H$ are left cosets $\{gH \mid g \in G\}$.

Proof: Suppose $G = \langle a \rangle$. Let $G/H$ be any factor group of $G$. WTS: $G/H$ is cyclic.

Any element of $G/H$ is of the form $gH$ for some $g \in G$.

Since $G$ is cyclic, there is an integer $i$ such that $g = a^i$. So $gH = a^iH$.

$a^iH = (aH)^i$ by definition of multiplication in factor groups.

Therefore $gH = (aH)^i$ for any coset $gH$.

So $G/H$ is cyclic, by definition of cyclic groups.

4. Prove that a factor group of an Abelian group is Abelian.

Answer: Recall: A group $G$ is Abelian if $ab = ba$ for all $a, b \in G$.

Proof: Suppose $G$ is Abelian. Let $G/H$ be any factor group of $G$. WTS: $G/H$ is Abelian.

Let $aH, bH$ be any elements of $G/H$.

Then $(aH)(bH) = (ab)H$ by definition of multiplication in factor groups.

Then $(ab)H = (ba)H$ since $G$ is abelian.

$(ba)H = (bH)(aH)$ again definition of multiplication in factor group.

Therefore $(aH)(bH) = (bH)(aH)$.

Therefore $G/H$ is Abelian.
5. Prove that any subgroup of an Abelian group is normal subgroup.

**Answer:** Recall: A subgroup \( H \) of a group \( G \) is called normal if \( gH = Hg \) for every \( g \in G \).

**Proof:** Suppose \( G \) is Abelian. Let \( H \) be a subgroup of \( G \). WTS: \( gH = Hg \) for all \( g \in G \).

Let \( g \in G \). Then \( gH = \{ gh | h \in H \} \) by definition of left coset.

\[ gh = hg \text{ for all } h \text{ since } G \text{ is Abelian.} \]

Therefore \( \{ gh | h \in H \} = \{ hg | h \in H \} = Hg \) by definition of right coset \( Hg \).

Therefore \( gH = Hg \) for all \( g \in G \). Therefore \( H \) is normal subgroup of \( G \). \( \square \)

6. Let \( H = \langle 5 \rangle \) be the subgroup of \( G = \mathbb{Z} \) generated by 5.

(a) List the elements of \( H = \langle 5 \rangle \).

**Answer:** Notice: Operation in \( G = \mathbb{Z} \) is addition.

\( H = \langle 5 \rangle = \{0, \pm 5, \pm 10, \ldots \} = \{5k | k \in \mathbb{Z}\} \)

(b) Prove that \( H \) is normal subgroup of \( G \).

**Answer:** Addition in \( \mathbb{Z} \) is commutative. So \( G = (\mathbb{Z}, +) \) is Abelian group and by previous problem every subgroup of an Abelian group is normal. \( \square \)

(c) List the elements of the factor group \( G/H = \mathbb{Z}/\langle 5 \rangle \).

**Answer:** Elements of \( G/H \) are left cosets of \( H \) in \( G \):

\[
0 + H = 0 + \langle 5 \rangle = \{0 + 0, 0 \pm 5, 0 \pm 10, \ldots \} = \{0, \pm 5, \pm 10, \ldots \} = H
\]

\[
1 + H = 1 + \langle 5 \rangle = \{1 + 0, 1 \pm 5, 1 \pm 10, \ldots \} = \{\ldots , -9, -4, 1, 6, 11, \ldots \}
\]

\[
2 + H = 2 + \langle 5 \rangle = \{2 + 0, 2 \pm 5, 2 \pm 10, \ldots \} = \{\ldots , -8, -3, 2, 7, 12, \ldots \}
\]

\[
3 + H = 3 + \langle 5 \rangle = \{3 + 0, 3 \pm 5, 3 \pm 10, \ldots \} = \{\ldots , -7, -2, 3, 8, 13, \ldots \}
\]

\[
4 + H = 4 + \langle 5 \rangle = \{4 + 0, 4 \pm 5, 4 \pm 10, \ldots \} = \{\ldots , -6, -1, 4, 9, 14, \ldots \}
\]

(d) Write the Cayley table of \( G/H = \mathbb{Z}/\langle 5 \rangle \).

\[
\begin{array}{cccccc}
G/H = \mathbb{Z}/\langle 5 \rangle & 0 + \langle 5 \rangle & 1 + \langle 5 \rangle & 2 + \langle 5 \rangle & 3 + \langle 5 \rangle & 4 + \langle 5 \rangle \\
0 + \langle 5 \rangle & 0 + \langle 5 \rangle & 1 + \langle 5 \rangle & 2 + \langle 5 \rangle & 3 + \langle 5 \rangle & 4 + \langle 5 \rangle \\
1 + \langle 5 \rangle & 1 + \langle 5 \rangle & 2 + \langle 5 \rangle & 3 + \langle 5 \rangle & 4 + \langle 5 \rangle & 0 + \langle 5 \rangle \\
2 + \langle 5 \rangle & 2 + \langle 5 \rangle & 3 + \langle 5 \rangle & 4 + \langle 5 \rangle & 0 + \langle 5 \rangle & 1 + \langle 5 \rangle \\
3 + \langle 5 \rangle & 3 + \langle 5 \rangle & 4 + \langle 5 \rangle & 0 + \langle 5 \rangle & 1 + \langle 5 \rangle & 2 + \langle 5 \rangle \\
4 + \langle 5 \rangle & 4 + \langle 5 \rangle & 0 + \langle 5 \rangle & 1 + \langle 5 \rangle & 2 + \langle 5 \rangle & 3 + \langle 5 \rangle \\
\end{array}
\]

(e) What is the order of \( 2 + \langle 5 \rangle \) in \( \mathbb{Z}/\langle 5 \rangle \)?

**Answer:** \( |2 + \langle 5 \rangle | = 5 \) as \( 2 + \langle 5 \rangle \in \mathbb{Z}/\langle 5 \rangle \).
7. Let $K = \langle 15 \rangle$ be the subgroup of $G = \mathbb{Z}$ generated by 15.

(a) List the elements of $K = \langle 15 \rangle$.
Answer: $K = \langle 15 \rangle = \{15k \mid k \in \mathbb{Z}\}$

(b) Prove that $K$ is normal subgroup of $G$.
Proof: $(\mathbb{Z} +)$ is Abelian group and any subgroup of an Abelian group is normal (from 5).

(c) List the elements of the factor group $G/K = \mathbb{Z}/\langle 15 \rangle$.
Answer: $G/K = \mathbb{Z}/\langle 15 \rangle = \{i + \langle 15 \rangle \mid 0 \leq i \leq 14\}$. (There are 15 elements.)

!!! This is just one way of expressing these cosets. Notice that there are many ways of expressing the same coset, eg. $(2 + \langle 15 \rangle) = (17 + \langle 15 \rangle) = (32 + \langle 15 \rangle) = (-13 + \langle 15 \rangle) = \ldots$.

(d) Write the Cayley table of $G/K = \mathbb{Z}/\langle 15 \rangle$.
Answer: Don’t have to actually write it, but make sure that you know what it would look like.

(e) What is the order of $3 + K = 3 + \langle 15 \rangle$ in $\mathbb{Z}/\langle 15 \rangle$?
Answer: $|3 + \langle 15 \rangle| = 5$ in $\mathbb{Z}/\langle 15 \rangle$, since
$(3 + \langle 15 \rangle) + (3 + \langle 15 \rangle) + (3 + \langle 15 \rangle) + (3 + \langle 15 \rangle) + (3 + \langle 15 \rangle) = (15 + \langle 15 \rangle) = (0 + \langle 15 \rangle)$.

(f) What is the order of $4 + K = 4 + \langle 15 \rangle$ in $\mathbb{Z}/\langle 15 \rangle$?
Answer: $|4 + \langle 15 \rangle| = 15$ in $\mathbb{Z}/\langle 15 \rangle$.

(g) What is the order of $5 + K = 5 + \langle 15 \rangle$ in $\mathbb{Z}/\langle 15 \rangle$?
Answer: $|5 + \langle 15 \rangle| = 3$ in $\mathbb{Z}/\langle 15 \rangle$.

(h) What is the order of $6 + K = 6 + \langle 15 \rangle$ in $\mathbb{Z}/\langle 15 \rangle$?
Answer: $|6 + \langle 15 \rangle| = 5$ in $\mathbb{Z}/\langle 15 \rangle$.

(i) Prove that $G/K$ is cyclic.
Answer: $\mathbb{Z}/\langle 15 \rangle$ is generated by $1 + \langle 15 \rangle$, hence it is cyclic.

underlineAnswer 2: $\mathbb{Z}/\langle 15 \rangle$ is cyclic since it is factor of the cyclic group $(\mathbb{Z}, +)$ (this group is generated by 1).
(j) Prove that $G/K = \mathbb{Z}/\langle 15 \rangle$ is isomorphic to $\mathbb{Z}_{15}$.

Answer:

- Elements of $\mathbb{Z}/\langle 15 \rangle$ are left cosets, and operation is addition of cosets (as defined for factor groups!)
- Elements of $\mathbb{Z}_{15}$: \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\} and addition is modulo 15.
- Define the map: $f : \mathbb{Z}/\langle 15 \rangle \to \mathbb{Z}_{15}$ by $f(i + \langle 15 \rangle) = r_i$, where $r_i$ is the remainder after dividing $i$ by 15.
  - Notice that $f$, as defined, is a mapping from $\mathbb{Z}/\langle 15 \rangle$ to $\mathbb{Z}_{15}$ since the values of $f$ are the remainders after dividing by 15, hence they are non-negative integers between 0 and 14.
  - Prove that $f$ is well-defined:
    - Suppose the same coset $(i + \langle 15 \rangle)$ is given by another integer $j$, i.e. $(i + \langle 15 \rangle) = (j + \langle 15 \rangle)$.
    - Then $f(i + \langle 15 \rangle) = r_i \in \mathbb{Z}_{15}$ and $f(j + \langle 15 \rangle) = r_j \in \mathbb{Z}_{15}$.
    - To show that $f$ is well defined we must show that $f(i + \langle 15 \rangle) = f(j + \langle 15 \rangle)$, i.e. WTS $r_i = r_j \in \mathbb{Z}_{15}$.
    - From $(i + \langle 15 \rangle) = (j + \langle 15 \rangle)$ it follows that $j \in (i + \langle 15 \rangle)$ and therefore $j = i + k \cdot 15$ for some integer $k \in \mathbb{Z}$.
    - By definition of remainders: $i = n \cdot 15 + r_i$ and $j = m \cdot 15 + r_j$, with $0 \leq r_i, r_j < 15$.
    - Therefore from $j = i + k \cdot 15$, we have $m \cdot 15 + r_j = n \cdot 15 + r_i + k \cdot 15$.
    - Hence $r_j - r_i = (k + n - m) \cdot 15$ which is an integer multiple of 15. Since the remainders are $0 \leq r_i, r_j < 15$, it follows that $r_j = r_i$.
    - Therefore $f$ is a well defined function.
  - Let $f : \mathbb{Z}/\langle 15 \rangle \to \mathbb{Z}_{15}$ defined by $f(i + \langle 15 \rangle) = r_i$, where $r_i$ is the remainder after dividing $i$ by 15. Then $f$ is ”one-to-one” function.
    - Proof: Suppose there are $a, b \in \mathbb{Z}/\langle 15 \rangle$ such that $f(a) = f(b)$. WTS $a = b$.
    - From $a, b \in \mathbb{Z}/\langle 15 \rangle$, it follows that $a = i + \langle 15 \rangle$ and $b = j + \langle 15 \rangle$.
    - From $f(a) = f(b)$ it follows $f(i + \langle 15 \rangle) = f(j + \langle 15 \rangle)$, and by definition of $f$ it follows that $r_i = r_j$.
    - So $i = n \cdot 15 + r_i$ and $j = m \cdot 15 + r_j$. Therefore $i - j$ is a multiple of 15, hence and element of the subgroup $\langle 15 \rangle$.
    - Therefore $a = i + \langle 15 \rangle = j + \langle 15 \rangle = b$.
    - Therefore $f$ is a one-to-one function.
  - $f : \mathbb{Z}/\langle 15 \rangle \to \mathbb{Z}_{15}$ is onto.
    - Proof: Let $y \in \mathbb{Z}_{15}$. WTS: there is an $x \in \mathbb{Z}/\langle 15 \rangle$ such that $f(x) = y$.
    - Since $y \in \mathbb{Z}_{15}$, $y$ is an integer $0 \leq y < 15$.
    - Let $x = y + \langle 15 \rangle \in \mathbb{Z}/\langle 15 \rangle$.
    - Then $f(x) = f(y + \langle 15 \rangle) = r_y$. Since $0 \leq y < 15$, it follows that $r_y = y$.
    - Therefore $f(x) = y$.
    - Therefore $f$ is onto.
- Therefore $f : \mathbb{Z}/\langle 15 \rangle \to \mathbb{Z}_{15}$ is an isomorphism. □
8. Let $G = \langle 6 \rangle$ and $H = \langle 24 \rangle$ be subgroups of $\mathbb{Z}$. Show that $H$ is a normal subgroup of $G$. Write the cosets of $H$ in $G$. Write the Cayley table for $G/H$.

$G$ and $H$ are subgroups of $\mathbb{Z}$, hence operation is addition.

$G = \langle 6 \rangle = \{0, \pm 6, \pm 12, \pm 18, \pm 24, \pm 30, \pm 36, \pm 42, \pm 48, \ldots \} = \{6j \mid j \in \mathbb{Z}\}$, multiples of 6.

$H = \langle 24 \rangle = \{0, \pm 24, \pm 48, \ldots \} = \{24j \mid j \in \mathbb{Z}\}$, i.e. all integer multiples of 24.

- $H$ is a normal subgroup of $G$.
  - $H$ is a nonempty subset of $G$, since elements of $H$ are elements of $G$ (integer multiples of 24 are multiples of 6).
  - $H$ is closed under operation: If $a, b \in H$, then $a = 24m$, and $b = 24n$. Therefore $a + b = 24m + 24n = 24(m + n) \in H$.
  - $H$ is closed under inverses: If $a \in H$, then $a = 24m$. Therefore $-a = 24 \cdot (-m)$, hence $-a \in H$.
  - Since $\mathbb{Z}$ is abelian, $G$ is also abelian and therefore any subgroup of $G$ is normal. Hence $H$ is normal in $G$.

- Cosets of $H = \langle 24 \rangle$ in $G = \langle 6 \rangle$ are:
  - $H = 0 + H = 0 + \langle 24 \rangle = \{\ldots, -48, -24, 0, 24, 48, \ldots\}$,
  - $6 + H = 6 + \langle 24 \rangle = \{\ldots, -42, -18, 6, 30, 54, \ldots\}$,
  - $12 + H = 12 + \langle 24 \rangle = \{\ldots, -36, -12, 12, 36, 60, \ldots\}$,
  - $18 + H = 18 + \langle 24 \rangle = \{\ldots, -30, -6, 18, 42, 66, \ldots\}$.

- Elements of $G/H$ are the four cosets written above and the Cayley table is:

<table>
<thead>
<tr>
<th>$G/H = \langle 6 \rangle / \langle 24 \rangle$</th>
<th>$0 + \langle 24 \rangle$</th>
<th>$6 + \langle 24 \rangle$</th>
<th>$12 + \langle 24 \rangle$</th>
<th>$18 + \langle 24 \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 + \langle 24 \rangle$</td>
<td>$0 + \langle 24 \rangle$</td>
<td>$6 + \langle 24 \rangle$</td>
<td>$12 + \langle 24 \rangle$</td>
<td>$18 + \langle 24 \rangle$</td>
</tr>
<tr>
<td>$6 + \langle 24 \rangle$</td>
<td>$6 + \langle 24 \rangle$</td>
<td>$12 + \langle 24 \rangle$</td>
<td>$18 + \langle 24 \rangle$</td>
<td>$0 + \langle 24 \rangle$</td>
</tr>
<tr>
<td>$12 + \langle 24 \rangle$</td>
<td>$12 + \langle 24 \rangle$</td>
<td>$18 + \langle 24 \rangle$</td>
<td>$0 + \langle 24 \rangle$</td>
<td>$6 + \langle 24 \rangle$</td>
</tr>
<tr>
<td>$18 + \langle 24 \rangle$</td>
<td>$18 + \langle 24 \rangle$</td>
<td>$0 + \langle 24 \rangle$</td>
<td>$6 + \langle 24 \rangle$</td>
<td>$12 + \langle 24 \rangle$</td>
</tr>
</tbody>
</table>

9. Viewing $\langle 6 \rangle$ and $\langle 24 \rangle$ as subgroups of $\mathbb{Z}$, prove that $\langle 6 \rangle / \langle 24 \rangle$ is isomorphic to $\mathbb{Z}_4$.

Proof: Elements of $\langle 6 \rangle / \langle 24 \rangle$ are the 4 cosets: $\{(0 + \langle 24 \rangle), (6 + \langle 24 \rangle), (12 + \langle 24 \rangle), (18 + \langle 24 \rangle)\}$ with the above multiplication table.

$|6 + \langle 24 \rangle| = 4$ since $(6 + \langle 24 \rangle) + (6 + \langle 24 \rangle) + (6 + \langle 24 \rangle) + (6 + \langle 24 \rangle) = (0 + \langle 24 \rangle)$.

So $\langle 6 \rangle / \langle 24 \rangle$ can be generated by one element of order 4, therefore it is cyclic of order 4.

The group $\mathbb{Z}_4$ is also cyclic of order 4. By theorem: "Any two cyclic groups of the same order are isomorphic.", it follows that $\langle 6 \rangle / \langle 24 \rangle$ is isomorphic to $\mathbb{Z}_4$. 


10. Let \( \langle 8 \rangle \) be the subgroup of \( \mathbb{Z}_{48} \).

(a) What is the order of the factor group \( \mathbb{Z}_{48}/\langle 8 \rangle \)?

Answer: Elements of \( \mathbb{Z}_{48} \) are \( \{0, 1, 2, 3, \ldots, 46, 47\} \) and \( |\mathbb{Z}_{48}| = 48 \).
Elements of \( \langle 8 \rangle \subset \mathbb{Z}_{48} \) are \( \{0, 8, 16, 24, 32, 40\} \) and \( |\langle 8 \rangle| = 6 \).
Therefore the order of the factor group is: \( |\mathbb{Z}_{48}/\langle 8 \rangle| = \frac{48}{6} = 8 \).

(b) What is the order of \( 2 + \langle 8 \rangle \) in the factor group \( \mathbb{Z}_{48}/\langle 8 \rangle \)?

Answer: Elements of \( \mathbb{Z}_{48}/\langle 8 \rangle \) are the following cosets:
\[ \{ (0 + \langle 8 \rangle), (1 + \langle 8 \rangle), (2 + \langle 8 \rangle), (3 + \langle 8 \rangle), (4 + \langle 8 \rangle), (5 + \langle 8 \rangle), (6 + \langle 8 \rangle), (7 + \langle 8 \rangle) \} \].
The order of \( (2 + \langle 8 \rangle) \) is \( |(2 + \langle 8 \rangle)| = 4 \) since 4 is the smallest number of times \( (2 + \langle 8 \rangle) \) must be added to itself in order to get the identity, i.e. such that \( (2 + \langle 8 \rangle) + (2 + \langle 8 \rangle) + (2 + \langle 8 \rangle) + (2 + \langle 8 \rangle) = (0 + \langle 8 \rangle) \).

11. Let \( G = U(16) \) be the group of units modulo 16.

(a) What is the order of \( G \)?
\[ G = \{1, 3, 5, 7, 9, 11, 13, 15\} \]. So \( |G| = 8 \).
We also know that \( |U(16)| = \phi(16) = \phi(2^4) = 2^4(2 - 1) = 8 \).

(b) What is the order of \( 15 \in U(16) \)?
\[ 15 \cdot 15 = 225 \equiv 1 \pmod{16} \]. Therefore \( |15| = 2 \) in \( U(16) \).
(Another way: \( 15 \cdot 15 = (-1)(-1) = 1 \equiv 1 \pmod{16} \))

(c) Let \( H = \langle 15 \rangle \) be the subgroup of \( U(16) \) generated by 15. What is the order of the factor group \( U(16)/H \)?
\( |G| = 8, |H| = 2 \). Therefore \( |G/H| = |G|/|H| = 8/2 = 4 \)
Also, in more details: \( H = \langle 15 \rangle = \{1, 15\} \).
Elements of \( G/H \) are the left cosets of \( H \) in \( G \):
\[ 1H = \{1, 15\} = 15H, \]
\[ 3H = \{3, 13\} = 13H \]
\[ 5H = \{5, 11\} = 11H \]
\[ 7H = \{7, 9\} = 9H \]

(d) Make the Cayley table of the factor group \( G/H = U(16)/\langle 15 \rangle \):

\[
\begin{array}{c|cccc}
G/H & 1H & 3H & 5H & 7H \\
\hline
1H & 1H & 3H & 5H & 7H \\
3H & 3H & 7H & 1H & 5H \\
5H & 5H & 1H & 7H & 3H \\
7H & 1H & 5H & 3H & 1H \\
\end{array}
\]