1. Consider the polynomial \( p(x) = x^2 + x + 2 \in \mathbb{Z}_3[x] \). Let \( K = \mathbb{Z}_3[x]/(x^2 + x + 2) \) be the congruence-class ring.

(a) Prove that \( K \) is a field.

(b) List all the elements of the field \( K \).

(c) How many elements are there in \( K \).

(d) Consider the polynomial ring \( K[x] \) and consider \( p(x) \) as an element of \( K[x] \). (Write \( p(x) \) with coefficients in \( K \).)

(e) Prove that the element \( [x] \in K \) is a root of the polynomial \( p(x) \in K[x] \).

(f) Prove that the element \( [2x + 2] \in K \) is a root of the polynomial \( p(x) \in K[x] \).

(g) Factor the polynomial \( p(x) = (x - a)(x - b) \in K[x], \) i.e. with \( a, b \in K \).

2. If \( p(x) \) is an irreducible quadratic polynomial in \( F[x] \), show that \( F[x]/(p(x)) \) contains all the roots of \( p(x) \).
3. Examples of rings and fields:

(a) Give an example of a finite ring which is not a field.

(b) Give an example of a finite ring which is a field.

(c) Give an example of an infinite ring which is not a field.

(d) Give an example of an infinite ring which is a field.

(e) Give an example of a ring with 4 elements which is not a field.

(f) Give an example of a ring with 4 elements which is a field.

(g) Give an example of a ring with 8 elements which is not a field.

(h) Give an example of a ring with 8 elements which is a field.

(i) Give two examples of a ring with 128 elements which is not a field.

(j) Give an example of a ring with 32 elements which is a field.

(k) Give an example of a ring with 9 elements which is not a field.

(l) Give an example of a ring with 9 elements which is a field.

(m) Give an example of a ring (which is not a field) which contains \( \mathbb{Z}_5 \) as a subring.

(n) Give an example of a field which contains \( \mathbb{Z}_5 \) as a subring.
4. Rings $R$ without identity $1_R$.

(a) Show that $3\mathbb{Z} = \{0, \pm3, \pm6, \ldots\}$ with addition and multiplication from $\mathbb{Z}$ is a ring without identity.

(b) Show that $n\mathbb{Z} = \{0, \pm n, \pm 2n, \ldots\}$ with addition and multiplication from $\mathbb{Z}$ is a ring without identity, for $n \neq \pm 1$.

(c) Let $R$ be a ring. Show that any ideal $I \subseteq R$ with addition and multiplication from $R$ is a ring. (This is usually ring without identity.)

(d) Give an example of a ring $R$ and an ideal $I \subseteq R$ with addition and multiplication from $R$ which is a ring with identity.

(e) Let $R$ be a ring. Show that any left ideal $I \subseteq R$ with addition and multiplication from $R$ is a ring. (This is usually ring without identity.)

(f) Give an example of a non-commutative ring $R$ and a left ideal $I \subseteq R$ with addition and multiplication from $R$ which is a ring with identity.

(g) Let $R$ be a ring. Show that any right ideal $I \subseteq R$ with addition and multiplication from $R$ is a ring.

(h) Give an example of a commutative ring and a subring which is not an ideal.

(i) Give an example of a non-commutative ring and a subring which is neither left nor right ideal.
5. Ring homomorphisms

(a) Let $R$ be a ring. Then $f : R \to R$ given by $f(x) = x$ is a ring homomorphism. What is $\text{Im}(f)$? What is $\text{Ker}(f)$?

(b) Let $R$ be a ring. Then $f : R \to R$ given by $f(x) = 0_R$ is a ring homomorphism. What is $\text{Im}(f)$? What is $\text{Ker}(f)$?

(c) Let $R, R'$ be a ring. Then $f : R \to R'$ given by $f(x) = 0_{R'}$ is a ring homomorphism. What is $\text{Im}(f)$? What is $\text{Ker}(f)$?

(d) Let $R$ be a ring and $I$ and ideal. Then $\pi : R \to R/I$ given by $\pi(r) = r + I$ is a ring homomorphism. What is $\text{Im}(\pi)$? What is $\text{Ker}(\pi)$?

(e) Consider the map $f : \mathbb{Z} \to \mathbb{Z}_8$ defined as $f(a) := [a]_8$, the congruence class of $a$ mod 8.
   i. What is $\text{Im}(f)$?
   ii. What is $\text{Ker}(f)$?
   iii. Prove that $\mathbb{Z}/(8\mathbb{Z}) \cong \mathbb{Z}_8$

(f) Consider the map $f : \mathbb{Z} \to \mathbb{Z}_n$ defined as $f(a) := [a]_n$, the congruence class of $a$ mod $n$.
   i. What is $\text{Im}(f)$?
   ii. What is $\text{Ker}(f)$?
   iii. Prove that $\mathbb{Z}/(n\mathbb{Z}) \cong \mathbb{Z}_n$

(g) Consider the map $f : \mathbb{Z}_{24} \to \mathbb{Z}_8$ defined as $f([n]_{24}) := [n]_8$, the congruence classes of $n$ mod 24 and 8 (respectively).
   i. Prove that $f$ is well defined.
   ii. Prove that $f$ is ring homomorphism.
   iii. What is $\text{Im}(f)$?
   iv. What is $\text{Ker}(f)$?
   v. Apply the First Isomorphism Theorem - write the isomorphism.
   vi. Consider the ideals $24\mathbb{Z} \subset 8\mathbb{Z} \subset \mathbb{Z}$. Apply the Third Isomorphism Theorem.
6. More ring homomorphisms

(a) Let \( R \) and \( S \) be rings and \( R \times S \) product ring. Prove that the map \( f : R \rightarrow R \times S \) given by \( f(r) = (r, 0_S) \) is a ring homomorphism. What is \( \text{Im}(f) \)? What is \( \text{Ker}(f) \)?

(b) Prove that the map \( f : \mathbb{Z}_8 \rightarrow \mathbb{Z}_8 \times \mathbb{Z}_3 \) given by \( f([a]_8) = ([a]_8, [0]_3) \) is a ring homomorphism. What is \( \text{Im}(f) \)? What is \( \text{Ker}(f) \)?

(c) Prove that the map \( f : \mathbb{Z}_8 \rightarrow \mathbb{Z}_{24} \) given by \( f([a]_8) = [9a]_{24} \) is a ring homomorphism. What is \( \text{Im}(f) \)? What is \( \text{Ker}(f) \)?

(d) Prove that the map \( f : \mathbb{Z} \rightarrow \mathbb{Z}_{24} \) given by \( f(a) = [9a]_{24} \) is a ring homomorphism. What is \( \text{Im}(f) \)? What is \( \text{Ker}(f) \)?

7. Some maps which are NOT ring homomorphisms. Prove it.

(a) \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) given by \( f(a) = 3a \) is not a ring homomorphism. Is it a group homomorphism?

(b) \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) given by \( f(a) = 3 + a \) is not a ring homomorphism. Is it a group homomorphism?

(c) \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) given by \( f(a) = na \) is not a ring homomorphism \((n \neq 1)\). Is it a group homomorphism?

(d) \( f : \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z} \) given by \( f(a) = (a, 1) \) is not a ring homomorphism. Is it a group homomorphism?

(e) \( f : \mathbb{Z} \rightarrow \mathbb{Z}_4 \) given by \( f(a) = [3a]_4 \) is not a ring homomorphism. Is it a group homomorphism?

8. Let \( R \) be a ring with \( 1_R \). Let \( f : R \rightarrow S \) be a ring homomorphism. Then:

(a) \( f(1_R) \) is an idempotent in \( S \). \( (e \) is an idempotent if \( e^2 = e) \)

(b) \( f(1_R) \) is an identity in the ring \( \text{Im}(f) \)
9. Which of the following maps are ring homomorphisms?

(a) \( f : \mathbb{Z} \to \mathbb{Q} \) given by \( f(a) = a \)

(b) \( f : \mathbb{Z} \to \mathbb{Q} \) given by \( f(a) = 2a \)

(c) \( f : \mathbb{Z} \to \mathbb{Z}_5 \) given by \( f(a) = [a]_5 \)

(d) \( f : \mathbb{Z} \to \mathbb{Z}_5 \) given by \( f(a) = [2a]_5 \)

(e) \( f : \mathbb{Z} \to \mathbb{Z}_6 \) given by \( f(a) = [a]_6 \)

(f) \( f : \mathbb{Z} \to \mathbb{Z}_6 \) given by \( f(a) = [2a]_6 \)

(g) \( f : \mathbb{Z} \to \mathbb{Z}_6 \) given by \( f(a) = [3a]_6 \)

(h) \( f : \mathbb{Z} \to \mathbb{Z}_6 \) given by \( f(a) = [4a]_6 \)

(i) \( f : \mathbb{Z} \to \mathbb{Z}_6 \) given by \( f(a) = [5a]_6 \)

(j) \( f : \mathbb{Z}_3 \to \mathbb{Z}_6 \) given by \( f([a]_3) = [a]_6 \)

(k) \( f : \mathbb{Z}_3 \to \mathbb{Z}_6 \) given by \( f([a]_3) = [2a]_6 \)

(l) \( f : \mathbb{Z}_3 \to \mathbb{Z}_2 \times \mathbb{Z}_3 \) given by \( f([a]_3) = ([0]_2, [a]_3) \)

(m) \( f : \mathbb{Z}_3 \to \mathbb{Z}_2 \times \mathbb{Z}_3 \) given by \( f([a]_3) = ([1]_2, [a]_3) \)

(n) \( f : \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_3 \) given by \( f([a]_3) = ([a]_2, [2]_3) \)