INTRODUCTION TO CLUSTER ALGEBRAS

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Cluster algebras are a class of commutative ring, introduced in 2000 by Fomin and Zelevinsky, originally to study Lusztig’s dual canonical basis and total positivity. After that, connections have been made to many other fields, including coordinate rings of Grassmannians, quiver representations, Teichmüller theory, invariant theory, tropical calculus, combinatorics, etc. In this introduction, we will start from the example of pentagon recurrence. Then we will give the definition of cluster algebra and state the Laurent phenomenon and the positivity conjecture. We will then explain the example of triangulation of an n-gon (coordinate ring of the grassmannian Gr(2,n)). Finally, the connection with quiver mutations will be mentioned. This introduction is based on Lauren Williams’ 1st lecture in a summer school on cluster algebras at MSRI in August, 2011.

1. Example: Pentagon Recurrence

Let us first give a combinatorial example. Consider a sequence \( f_1, f_2, \ldots \), defined recursively by the following relations:

\[
\begin{align*}
  f_1 & : = x \\
  f_2 & : = y \\
  f_{n+1} & = \frac{f_n + 1}{f_{n-1}}
\end{align*}
\]

A few of the values:

\[
\begin{array}{cccccccc}
  f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & \cdots \\
  x & y & \frac{y+1}{x} & \frac{y+1+x}{xy} & \frac{x+1}{y} & x & y & \cdots
\end{array}
\]

(the sequence will now be periodic)

Two observations:

i. From the recurrence, it is not clear that the denominator should be a monomial (surprising cancelation). Clearly we will get rational functions, but in fact we get Laurent polynomials (in the initial variables).

ii. In the numerator, the coefficients are non-negative integers.
Let us rewrite this example using matrices:

\[
\begin{bmatrix}
  f_1 \\
  f_2 \\
\end{bmatrix}
\xrightarrow{\mu_1}
\begin{bmatrix}
  f_3 \\
  f_2 \\
\end{bmatrix}
\xrightarrow{\mu_2}
\begin{bmatrix}
  f_3 \\
  f_4 \\
\end{bmatrix}
\xrightarrow{\mu_1}
\begin{bmatrix}
  f_3 \\
  f_4 \\
\end{bmatrix}
\xrightarrow{\mu_2}
\ldots
\]

and we call the relation a mutation relation.

This is a very simple example of a cluster algebra. In fact, we shall see that it is the cluster algebra related to the Grassmannian $Gr(2, 4)$.

\section{2. Formal Definition of Cluster Algebras}

Start with a \textit{seed}, i.e., the data $X = \{x_1, \ldots, x_n\}, B$ consisting of a collection of $n$ algebraically independent variables, and an $n \times n$ skew-symmetrizable integer matrix $B$.

From this seed, we can \textit{mutate} in each of $n$ directions to obtain $n$ more seeds. The mutation is defined by $B$. That is, columns of the matrix $B$ encode the exchange relation in the following way. For $k \in \{1, \ldots, n\}$,

\[x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}\]

So the new seed $\mu_k((X, B)) = \{x_1, \ldots, \hat{x}_k, \ldots, x_n\} \cup \{x'_k\}, \mu_k(B))$ where

\[(\mu_k(B))_{ij} = \begin{cases} 
-b_{ij} & \text{if } k = i \text{ or } k = j \\
b_{ij} & \text{if } b_{ik} b_{kj} \leq 0 \\
b_{ij} + b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} > 0 \\
b_{ij} - b_{ik} b_{ij} & \text{if } b_{ik}, b_{kj} < 0
\end{cases}\]

One can check: $\mu_k(B)$ is still skew-symmetrizable. Starting from an initial seed, one can apply all possible sequences of mutations. This gives a set (possibly infinite) of all cluster variables.

\textbf{Definition 2.1.} The cluster algebra $A(X, B)$ is the subalgebra of $k(x_1, \ldots, x_n)$ generated by all cluster variables (together with the data of the clusters, meaning the clusters).

\textbf{Example 2.2.} The matrix of $1$ is precisely $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

\textbf{2.1. Notable Notes.}

\textbf{Theorem 2.3} (Fomin-Zelevinsky). The cluster variables are always Laurent Polynomials with coefficients in $\mathbb{Z}$.

\textbf{Conjecture 2.4} (Fomin-Zelevinsky). Any cluster variables is $p(x_1, \ldots, x_n)/m(x_1, \ldots, x_n)$ with $p(x_1, \ldots, x_n) \in \mathbb{N}[x_1, \ldots, x_n]$. 
Consider a polygon with \( n+3 \) sides, choose any triangulation \( T \) (will always have \( n+3 \) boundary segments and \( n \) diagonals).

The set of all triangulations of an \( n+3 \)-gon are connected by elementary flips.

\[
\begin{array}{c}
\text{can be flipped to}
\end{array}
\]

**Example 3.1.** For \( n = 2 \), we have the following triangulations and mutations:

**Example 3.2.** Associate an \( n \times n \) matrix \( B(T) \) to a triangulation \( T \) as follows: label the \( n \) diagonals of \( T \) from \( 1, \ldots, n \). Take \( B(T)_{ij} = \# \{ \text{triangles with sides } i \text{ and } j, \text{ with } j \text{ following } I \text{ in clockwise order} \} - \# \{ \text{triangles with sides } i \text{ and } j \text{ with } j \text{ following } i \text{ in counterclockwise order} \} \). So, for the triangulation of the hexagon,

\[
B(T) = \begin{bmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{bmatrix}
\]

Notice: \( \mu_1(B(T)) = B(\mu_1(T)) \).

We have the following bijections:

- Cluster variables \( x_\gamma \) correspond to the diagonal \( \gamma \) in some triangulation.
- Every cluster corresponds to a triangulation
• The exchange relation corresponds to flips

Namely, the exchange relation in a square \( \alpha, \beta, \gamma, \delta \) with diagonal \( \Theta \), and new diagonal \( \eta \) is given by \( x_\Theta x_\eta = x_\alpha x_\gamma + x_\beta x_\delta \). But sometimes we would like to assign a variable to the boundary segments in the polygon. In this way we get a cluster algebra with so-called \textit{frozen} variables (sometimes called coefficients).

4. \textsc{coordinate ring of the grassmannian } \textit{Gr}(2, n + 3)

The homogeneous coordinate ring of \( \textit{Gr}(2, n + 3) \) is given by

\[
k[p_{ij} \mid 1 \leq i < j \leq n + 3]/ \langle \text{plucker relations} \rangle
\]

where the plucker relations are of the following form: if \( i < j < k < l \), then \( p_{ik}p_{jl} = p_{ij}p_{lk} + p_{il}p_{jk} \). This can be viewed in a square with vertices (going clockwise) \( i, j, k, l \). The coordinate ring of \( \textit{Gr}(2, n + 3) \) has a structure of a cluster algebras (with coefficients) such that

• The coefficients are the variables \( p_{i,i+1} \) (associated to the sides of the \( n + 3 \) gon);
• The cluster variables are the variables corresponding to the diagonals;
• The clusters are \( n \)-tuples of variables whose corresponding diagonals form a triangulation;
• The exchange relations are the plucker relations, and the cluster type is \( A_n \) (to be described in a moment).

The \( n = 2 \) case corresponds to the pentagon recurrence from section \ref{pentagon}

5. \textsc{connection with quiver mutations.}

Starting from a seed \((\{x_1, \ldots, x_n\}, R)\), instead of skew-symmetrizable matrix, \( R \) is a finite quiver (digraph) on \( n \) vertices, and without loops or two-cycles. Mutation is defined by

\[
x_kx'_k = \prod_{a:i \rightarrow k} x_i + \prod_{a:k \rightarrow j} x_j
\]

and mutation for the quiver (at a vertex) goes like this:

i. Reverse all arrows incident with \( k \),
ii. for each 2-path \( ba \) through \( i \rightarrow k \rightarrow j \), add an arrow \( j \rightarrow i \),
iii. remove all 2-cycles

For example, applying \( \mu_1 \) to the quiver below:

\[
\begin{align*}
1 & \sim \sim
3 & \rightarrow 2 \\
& \sim
3 & \rightarrow 2
\end{align*}
\]
6. Finite Type

**Theorem 6.1** (Fomin-Zelevinsky). *The number of cluster variables in \( \mathcal{A}(X, R) \) is finite if and only if \( R \) is mutationally equivalent to an orientation of an ADE Dynkin diagram. Furthermore, all orientation of the same ADE Dynkin diagram are mutation equivalent. In this case, the set of cluster variables is in bijection with the set of almost positive roots, that is positive roots together with the negative simple roots.*

We note that many quivers which are not orientations of ADE Dynkin diagrams are mutation equivalent to such quivers, and therefore do define cluster algebras with finitely many cluster variables.

We note that there is also a cluster algebra associated to each non-simply laced Dynkin diagram, although these are not related to quivers in the way described above. Each of these cluster algebras has a finite number of cluster variables. In fact, this completes the classification of cluster algebras with finitely many cluster variables.

7. Questions that came up

(1) To what extend does the underlying subalgebra of \( k[x_1, \ldots, x_{mn}] \) determine the rest of the data (i.e. the clusters and the exchange relations)?

(2) What classical geometry related to the Grassmannian is being captured by the fact that there is a cluster algebra structure on its coordinate ring? (this should be related to positivity)