MODULI OF PRINCIPAL BUNDLES FOR A 2-GROUP:

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Plan for the talk

1) Some motivation: Dijkgraaf-Witten theory

2) Definitions: 2-groups & their principal bundles

3) Construction of a the Freed-Quinn line bundle.

§1. DIJKGRAAF-WITTEN THEORY:
Associated to a compact group $G$ and a 3-cocycle $\alpha \in Z^3(G;U(1))$
we define a 3d TQFT $T$ "Lagrangian"

- Chern-Simons theory for $G$ simple, simply connected
- Dijkgraaf-Witten theory for $G$ finite.

Data of a TQFT:

- to each closed 2-manifold $X$ $\mapsto T(X)$ a vector space $\emptyset \mapsto \mathbb{C}$

- to each 3-manifold $N$ with boundary $X_{\text{in}} \sqcup X_{\text{out}}$ $\mapsto T(N): T(X_{\text{in}}) \rightarrow T(X_{\text{out}})$

a linear map.

$\emptyset \quad \emptyset$

+ compositibilities
Physicists construct this data from \((G, \alpha)\) using path-integrals:

\[
M \leadsto E_M(G, \alpha) - \text{space of fields } \Theta \text{ on } M
\]

\[
d\mu_M - \text{measure on } M
\]

\[
S_X - \text{action on } M
\]

\[
\mapsto T(M) = \int d\mu_M(\Theta) e^{i\alpha S_X(\Theta)}
\]

"integrate" = "pushforward"

Result can be a number, vector space, sheaf, etc...

\[\square\] **Problem:** \(d\mu_M\) might not exist.

**Special case:** \(G\) finite, \(X\) a surface.

\[E_X = \{\text{equivalence classes of (flat) principal } G\text{-bundles}\} \text{ on } X\]

\[= \text{Fun}(\star//T_t(X), \star//G) / G\]

\[\square\] \(E_X\) is a finite set!

\[\sum_{C_x} \text{ is a finite sum!}\]

**Define:** \(T(X) := \text{space of sections of a certain } G\text{-equivariant line bundle } L \text{ on}\)

\[\text{Fun}(\star//T_t(X), \star//G)\]

\[\square\] \(L\) is called the Freed-Quinn line bundle

**Construction of \(L\):**

- To write down a line bundle, we need to glue a

\[
\text{functor } \text{Fun}(\star//T_t(X), \star//G) \longrightarrow \star//\text{U}(1).
\]
Inside \( \text{Fun}(\pi_1(X), \ast \to \mathbb{G}) \), we have a full, skeletal subcategory

\[
\underbrace{\ast \to \text{Aut}(g)}_{[g] \in \mathcal{C}_x}
\]

and it's enough to specify the functor there.

So we need to give functors

\[
\ast \overset{\text{Aut}(g)}{\longrightarrow} \ast \overset{\text{UC}(1)}{\longrightarrow}, \quad \text{equiv. group homomorphisms}
\]

\[
f_g : \text{Aut}(g) \longrightarrow \text{UC}(1).
\]

Recall that \( g : \pi_1(X) \longrightarrow \mathbb{G} \), and \( \text{Aut}(g) = \{ t \in \mathbb{G} \mid t g t^{-1} = g \} \)

* given \( t : \mathbb{P}_g \longrightarrow \mathbb{P}_g \) , we define a principal \( \mathbb{G} \)-bundle

on \( X \times S^1 \) by taking \( \mathbb{P}_g \times [0,1] \) and gluing \( \mathbb{P}_g \times \{0\} \) and \( \mathbb{P}_g \times \{1\} \) via \( t \).

\[ \Rightarrow \text{we obtain } g_t : X \times S^1 \longrightarrow \mathbb{B} \mathbb{G}. \]

* define \( f_g (t) := \int_{X \times S^1} g_t \ast \alpha \in \text{UC}(1). \)

\[ \Rightarrow \text{we thus obtain the line bundle } \mathcal{L}, \text{ classified by a } \]

2-cocycle obtained from \( \alpha \) via "transgression".

* One goal of this talk: provide a geometric/categorical

construction of the total space of \( \mathcal{L} \), starting with

the data of \( (G, \alpha) \).

* Strategy: Starting from the data of \( (G, \alpha) \), we construct

a "2-group" \( \mathcal{G} \).
• We study the moduli space \( M \) of "principal \( G \)-bundles" and observe that it defines a \( G \)-equivariant fibration \( M \) \[
\downarrow \\
\text{Bun}_G^k(X)
\]
• We impose an equivalence relation on \( M \) and show that the resulting quotient space is the total space of the Freed-Quinn line bundle.

82. 2-Groups

**Definition.** A weak 2-group \( G \) is a monoidal groupoid in which all objects and morphisms admit \( \otimes \)-inverses.

**Example:** Let \( H \) be any group.

1. Define \( H \) to be the 2-group with objects \( * \) and \( * \), only identity morphisms, and tensor product
   \[ g \otimes h \to g \otimes h. \]
   
   The associator \( (g \otimes h) \otimes k \to g \otimes (h \otimes k) \) must be
   
   \[ \text{id}_{g \otimes h \otimes k}. \]

2. Define \( * \) to be the 2-group with object \( * \),
   
   morphisms \( \text{Hom}(*,*) = H \).
   
   We want the tensor structure to be given by
   
   - \( * \otimes * = * \)
   - \( h \otimes k = hk \in H \)
   
   In order for this to work out, \( H \) must be abelian.
Example: Fix $G$ a finite group

$$\alpha \in Z^3(G; \text{ucr}) \quad \alpha \text{ is cocycle.}$$

$$\alpha : G \times G \times G \rightarrow \text{ucr}$$

Then define $G^\cdot$ to be the category with

- objects $g \in G$
- morphisms $\text{Hom}_G(g, h) = \begin{cases} \emptyset & g \neq h \\ \text{ucr} & g = h \end{cases}$

with $g \xrightarrow{\alpha} g \xrightarrow{b} g$

Define monoidal structure on $G^\cdot$:

- on objects $g \cdot h = gh$
- on morphisms:
  $$\alpha_{g,h} : g \cdot h \rightarrow g \cdot h$$
  $$\alpha_{g,h} \in \text{Hom}_G(gh, gh)$$
- unit object $1$
- associator:
  $$\alpha((g \cdot h) \cdot k) \xrightarrow{\sim} g \cdot (h \cdot k)$$
  $$\alpha(g, h, k) \cdot 1$$

Pentagon axiom: given $g, h, k, l$

$$(g \cdot h) \cdot (k \cdot l) \xrightarrow{\alpha(g, h, k, l)} g \cdot (h \cdot (k \cdot l))$$

$$
\begin{array}{ccc}
\alpha(g, h, k, l) & \xrightarrow{\sim} & g \cdot (h \cdot (k \cdot l)) \\
\alpha(g, h, k, l) & \xrightarrow{\sim} & g \cdot (h \cdot (k \cdot l)) \\
\alpha(g, h, k, l) & \xrightarrow{\sim} & g \cdot (h \cdot (k \cdot l)) \\
\alpha(g, h, k, l) & \xrightarrow{\sim} & g \cdot (h \cdot (k \cdot l)) \\
\end{array}
$$
\[ \alpha(gh,k,l) \alpha(g,hk,l) \alpha(g,h,k) = 1 \]

This is exactly the 3-cocycle condition.

**Fact:** if \( \alpha' = \alpha \circ \beta \) for \( \beta \) a 2-cocycle, then \((G, \alpha') \) & \((G, \alpha)\) give equivalent 2-groups.

So WLOG we can assume \( \alpha \) is normalised:

- \( \alpha(g,1,k) = 1 \) \( \forall g,k \).

\[ \Rightarrow \alpha(g,h,k) = 1 \text{ whenever } g,h, \text{ or } k = 1. \]

**Theorem** [Sinh, Baez-Lauda]

All essentially finite 2-groups arise in basically this way:

More precisely, any 2-group \( G \) with finitely many isomorphism classes of objects is determined by the data of:

- \( G \) a finite group
- \( A \) a \( G \)-module (we took the abelian group \( UC(1) \) with trivial \( G \)-action)
- a normalised 3-cocycle \( \alpha \in Z^3(G; A) \).

The 2-group \( G \) defined by \((G,A,\alpha)\) fits into a SES:

\[ 1 \rightarrow \mathbb{Z}/A \rightarrow \mathbb{Z} \rightarrow G \rightarrow 1 \]

- it is a central extension \( \iff \) \( A \) is a trivial \( G \)-module.

*(e.g. in our \( UC(1) \)-example)*

**Recall:** Central extensions of \( G \) by \( UC(1) \) are classified by \( H^2(G; UC(1)) \).

Analogously: central extensions of \( G \) by \( \mathbb{Z}/UC(1) \) are
§3. PRINCIPAL BUNDLES TDR \( E = (G, \alpha) \)

**Warm-up:** Fix \( G, H \) finite 2-groups with underlying finite groups \( G, H \).

A homomorphism \( \varphi: G \rightarrow H \) should be a weak-monoidal functor:

- on objects: \( \varphi_0: G \rightarrow H \)
- monoidal structure \( \mu(g, h): \varphi_0(g) \varphi_0(h) \xrightarrow{\sim} \varphi_0(gh) \)

\( H \) skeletal \( \Rightarrow \varphi_0(g) \varphi_0(h) = \varphi_0(gh) \) so \( \varphi_0 \) is a homomorphism

and \( \mu: G \times G \rightarrow \mathcal{U}(1) \) provides a family of isomorphisms manifesting the homomorphism structure.

**Principle:** When we lift from the group setting to the 2-group setting, we require all the usual data to satisfy all the usual equations,

AND we require a family of elements in \( \mathcal{U}(1) \) demonstrating that these equations hold.

**Recall:** A (flat) principal \( G \)-bundle is determined by a group homomorphism

\[ \varphi: \pi_1(X) \rightarrow G \]

\( \therefore \) a flat principal \( G \)-bundle is determined by a pair:

\[ (g, \varphi) : \overset{\circ}{\varphi}: \pi_1(X) \rightarrow G \]

\overset{\circ}{\varphi}: \pi_1(X) \times \pi_1(X) \rightarrow \mathcal{U}(1) \]
compatibility condition:

\[ g(a, b) \cdot g(c) \xrightarrow{\alpha(g(a), g(b), g(c))} g(a) \cdot (g(b) \cdot g(c)) \]

\[ g(a, b) \cdot g(c) \xrightarrow{g(a, b, c)} g(a \cdot b \cdot c) \xrightarrow{g(a, b, c)} g(a) \cdot g(b) \cdot g(c) \]

\[ \Rightarrow \alpha(g(a), g(b), g(c)) = g(a, b, c) \cdot g(a, b, c) \]

\[ \Rightarrow g^* \alpha = \frac{1}{d\gamma}. \]

- An isomorphism of flat \( G \)-bundles is

\[ \tau : (g, \gamma) \rightarrow (g', \gamma') \]

where \( \tau \in G \) s.t.

\[ \tau g(a) \tau^{-1} = g'(a) \quad \forall \ a \in \pi_1(X). \]

- An isomorphism of flat \( G \)-bundles is a pair

\[ \psi : (g, \gamma) \rightarrow (g', \gamma') \]

- \( \psi \in \text{Fun}(\pi_1(X), G) \)

\[ \psi(a) \quad \text{satisfies a natural compatibility condition involving} \]

\[ g, \gamma', \alpha. \]

- In fact, flat principal \( G \)-bundles form a 2-category

\[ \text{Bun}^b_G(X) = \text{Fun}(\pi_1(X), G_G) \]

\[ \xrightarrow{\psi(a)} g \xrightarrow{\psi(a)} \]

\[ \tau \]

\[ \text{Bun}^b_G(X) = \text{Fun}(\pi_1(X), G_G) \]
Theorem: \( \text{Bun}^b_g(X) \) admits a natural right \( G \)-action, such that it is \( G \)-equivariant.

- Recall action of \( G \) on category \( \text{Bun}^b_g(X) \):

\[
F: G \rightarrow \text{Aut}(\text{Bun}^b_g(X)) \quad \text{monoidal functor.}
\]

\[
g \mapsto F_g
\]

\[
F_g(s) = g^{-1}sg,
\]

\[
F_g(t) = g^{-1}tg.
\]

\[
F(g,h): F_g \circ F_h \Rightarrow F_{gh}
\]

+ associativity condition for \( g,h,k \)

- We want to lift it to an action on the 2-category

\[\overline{F}: G \rightarrow \text{Aut}(\text{Bun}^b_g(X))\]

\[
g \mapsto \overline{F}_g
\]

\[
\overline{F}(g,h): \overline{F}_g \circ \overline{F}_h \Rightarrow \overline{F}_{gh}
\]

+ associativity data for \( g,h,k \)

+ associativity condition for \( g,h,k,l \).

E.g., \( \overline{F}_g : (g,x) \mapsto (g^3, gx) \)

Here \( g^3 : \pi_1(X) \rightarrow G \)

\[
\gamma(a,b) : g(a)g(b) \rightarrow g(ab)
\]

\[
\gamma(a,b,c) : g^{-1}(g(a)c)g \rightarrow g^{-1}(g(ab)c)g
\]

\[
\gamma(a,b) : g(a)g(b) \rightarrow g(ab)
\]

\[
g^{-1}(g(a)c)g \rightarrow g^{-1}(g(ab)c)g
\]

\[
\gamma(a,b,c) : g^{-1}(g(a)c)g \rightarrow g^{-1}(g(ab)c)g
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\[
g^{-1}(g(a)c)g \rightarrow g^{-1}(g(ab)c)g
\]

\[
\gamma(a,b) : g(a)g(b) \rightarrow g(ab)
\]

\[
\gamma(a,b,c) : g^{-1}(g(a)c)g \rightarrow g^{-1}(g(ab)c)g
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g^{-1}(g(a)c)g \rightarrow g^{-1}(g(ab)c)g
\]

\[
\gamma(a,b) : g(a)g(b) \rightarrow g(ab)
\]

\[
\gamma(a,b,c) : g^{-1}(g(a)c)g \rightarrow g^{-1}(g(ab)c)g
\]

\[
g^{-1}(g(a)c)g \rightarrow g^{-1}(g(ab)c)g
\]
§ 4. CONSTRUCTING THE FREED-QUINN LINE BUNDLE.

What is the fibre of this functor $\Gamma$ over $g$?

\[ \{ (g, \gamma) \mid d\gamma = \frac{1}{g^*a} \} . \]

So if $\gamma, \gamma'$ are in the fibre, $d\gamma = d\gamma'$

\[ \Rightarrow \text{ they differ by a unique 2-cocyle.} \]

i.e. $\text{ Fib}_g$ is a $\mathbb{Z}^2(\pi_1(X); U(1))$

We take the quotient identifying $(g, \gamma), (g, \gamma')$ iff $\gamma/\gamma'$ is

a coboundary

by performing an "associated bundle" construction

\[ \text{Bun}_G^b(X) \xrightarrow{\mathbb{Z}^2 \times H^2} \text{ now has fibres } H^2(\pi_1(X); U(1)) \]

\[ \text{U}(1) \]

**Theorem:** This $U(1)$-bundle agrees with the Freed-Quinn
line bundle $\mathcal{L}$.

**Sketch of proof:**

\[ \begin{tikzcd}
\text{Bun}_G^b(X) \ar[r] \ar[dr] & U(1)/U(1) \\
\text{Bun}_G^b(X) \times H^2 \ar[u] \ar[r] & U(1)/U(1) \ar[u] \ar[ur]
\end{tikzcd} \]

**Key lemma:** \[ [g, \gamma] = [g, \gamma'] \iff \int_X \gamma = \int_X g^*a \]

\[ \text{FREED-QUINN} \]