MATH 1365 Intro to Mathematical Reasoning, Fall 2011, Quiz 6

1) Prove that $3 \cdot 2^n < n!$ for any integer $n > 4$.
   **Induction!**
   **Basis step:** $n = 5$.
   
   $96 = 3 \cdot 2^5 < 5! = 120$ True!

   **Inductive step:** assume $3 \cdot 2^k < k!$ for some $k \geq 5$. Need to show: $3 \cdot 2^{k+1} < (k+1)!$
   Know by assumption: $3 \cdot 2^k < (3 \cdot 2^k) \cdot 2 < 2k!$
   So it is enough to show: $2k! \leq (k+1)! = (k+1) \cdot k!$
   which is logically equivalent to $2 \leq k+1 \iff k \geq 1$ True for $k \geq 5$.

2) Let $a_0 = 3$ and, for $n > 0$, let $a_n = 5a_{n-1} - 2$. Find and prove an explicit formula for $a_n$.
   Look for a constant $c$ such that the sequence ($b_n = a_n + c$) satisfies
   $b_n = 5b_{n-1} \iff a_n + c = 5(a_{n-1} + c) \iff a_n = 5a_{n-1} + 4c$.
   Works for $4c = -2 \iff c = -\frac{1}{2}$. So
   $a_n = b_n + \frac{1}{2} = 5^n b_0 + \frac{1}{2} = 5^n (a_0 - \frac{1}{2}) + \frac{1}{2} = 5^n \frac{5}{2} + \frac{1}{2} = \frac{5^{n+1} + 1}{2}$

3) Let $F_0, F_1, \ldots$ be the Fibonacci numbers defined by $F_0 = F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Prove that $F_1 + F_3 + \ldots + F_{2n-1} = F_{2n} - 1$ for all $n \geq 1$.
   **Induction!**
   **Basis step:** $n = 1$.
   $F_1 = F_2 - 1$. True!

   **Inductive step:** assume $F_1 + F_3 + \ldots + F_{2k-1} = F_{2k} - 1$ for some $k \geq 1$. Need to show:
   $F_1 + F_3 + \ldots + F_{2k-1} + F_{2k+1} = F_{2k+2} - 1$. Enough to show: $F_{2k+2} - F_{2k+1} = F_{2k+2} - F_{2k+1} \iff F_{2k+1} = F_{2k+2} - F_{2k+1}$
   by definition of Fibonacci numbers.

4) Extra credit. Prove that every amount of postage of 8 cents or more can be formed using just 3-cent and 5-cent stamps.
   **Proof by contradiction using smallest counterexample.** Suppose for the sake of contradiction that $n \geq 8$ is the smallest postage that cannot be formed using 3-cent and 5-cent stamps. Since $8 = 5 + 3$, $9 = 3 + 3 + 3$, $10 = 5 + 5$, we must have $n \geq 11$. But then $n-3 \geq 8$, cannot be formed using 3- and 5-cent stamps (since $n-3 \leq n$). Since $n = (n-3) + 3$ we obtain a contradiction!