MATH 1365 Intro to Mathematical Reasoning, Fall 2011, Quiz 1

1) A pair \((p, q)\) of integers is called a twin prime pair if both \(p\) and \(q\) are primes, and \(q = p + 2\). List the first four twin prime pairs.

Primes: \(2, 3, 5, 7, 11, 13, 17, 19, \ldots\)

Twin prime pairs: \((3, 5), (5, 7), (11, 13), (17, 19), \ldots\)

2) Let \(a, b\) and \(c\) be integers.
   (a) Disprove the following by a counterexample: if \(c^2 \mid ab\), then \(c \mid a\) or \(c \mid b\).

\[
c = 6, \quad a = 4, \quad b = 9
\]

\[
36 = 6^2 \mid 4 \cdot 9 = 36, \quad \text{but} \quad 6 \nmid 4, \quad \text{and} \quad 6 \nmid 9
\]

(b) Prove that if \(a \mid c\) and \(b \mid c\), then \((ab) \mid c^2\).

Know: \(c = ax\), and \(c = by\), for some integers \(x\) and \(y\).

Need to prove: \(c^2 = abz\), for some integer \(z\).

Multiply two known equalities together to get

\[
c^2 = (ax)(by) = ab(x y) = abz, \quad \text{where} \quad z = xy
\]

is an integer. Done!

3) Prove that any integer divisible by 3 can be expressed as the sum of three consecutive integers.

By definition, \(3 \mid n\) means that \(n = 3x\) for some integer \(x\). Then

\[
n = (x-1) + x + (x+1),
\]

the sum of three consecutive integers.

4) (Bonus Problem) Let \(a\) and \(b\) be positive integers. How many positive divisors does the number \(2^a \cdot 3^b\) have? Justify your answer.

Answer: \((a+1)(b+1)\). Proof: positive divisors of \(2^a \cdot 3^b\) are precisely the numbers \(2^r \cdot 3^s\) with \(0 \leq r \leq a\), and \(0 \leq s \leq b\). Then \(r\) takes \((a+1)\) possible values, while \(s\) takes \((b+1)\) possible values. Combining them, we get our answer.