1. Prove that the Boolean expression $(x \rightarrow y) \rightarrow z$ is logically equivalent to the negation of $(x \lor z) \rightarrow (y \land (\neg z))$.

$$(x \rightarrow y) \rightarrow z = \neg (x \lor y) \rightarrow z = \neg \neg (x \lor y) \lor z = (x \land \neg y) \lor z = (x \lor z) \land (\neg y \lor z).$$

Its negation is $\neg (x \lor z) \lor (y \land \neg z) = (x \lor z) \rightarrow (y \land \neg z)$.

2. Let $A, B, C$ be sets. Prove that $A - (B \cap C) = (A - B) \cup (A - C)$.
3. How many ways are there to place 6 identical white pawns on a $8 \times 8$ chessboard so that they will occupy two horizontal rows, with 4 pawns in one of them, and 2 pawns in another?

$\binom{8}{2} = 8 \cdot 7$ ways to pick two occupied rows

$\binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2}$ ways to pick 4 places for pawns in the first row

$\binom{8}{2} = \frac{8 \cdot 7}{2}$ ways to pick 2 places for pawns in the second row.

Answer: $8 \cdot 7 \cdot \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2} \cdot \frac{8 \cdot 7}{2}$

4. Prove or disprove: if $a, b$ and $c$ are positive integers, then $\gcd(a, bc) = \gcd(a, b) \cdot \gcd(a, c)$.

Disprove by a counterexample: $a = b = c = 2$, then $\gcd(a, bc) = \gcd(2, 4) = 2$, but $\gcd(a, b) \cdot \gcd(a, c) = 2 \cdot 2 = 4$. 

5. What is the smallest positive integer \( n \) such that any string of length \( n \) consisting of letters \( a, b, \) and \( c \) must have a repeated consecutive subsequence of length 4? (In the string \( (a, b, c, a) \) the consecutive subsequences of length 2 are \( (a, b), (b, c), \) and \( (c, a) \).)

There are \( 3^4 = 81 \) possible sequences of length 4.
A string of length \( n \) has \( n-3 \) consecutive subsequences of length 4 (starting at any of the places \( 1, 2, \ldots, n-3 \)).
So if \( n-3 = 82 \) then by the Pigeonhole Principle, there will be a repeated subsequence of length 4.

Answer: \( n = 85 \)

6. Find the coefficient of \( x^5 \) in the binomial expansion of \( (x + \sqrt{x})^8 \).

Terms in the expansion: \( \binom{8}{k} \cdot x^{8-k} \cdot (\sqrt{x})^k = \binom{8}{k} \cdot x^{8-k} \cdot x^{k/2} = \binom{8}{k} \cdot x^{8-\frac{k}{2}} \)

Need to find \( k \) so that \( 8 - \frac{k}{2} = 5 \iff k = 6 \).

Answer: \( \binom{8}{6} = \binom{8}{2} = \frac{8 \cdot 7}{2} = 28 \).
7. Prove by mathematical induction that
\[
\binom{3}{3} + \binom{4}{3} + \cdots + \binom{n}{3} = \binom{n+1}{4}
\]
for all integers \( n \geq 3 \).

**Basis step:** \( n = 3 \): \( \binom{3}{3} \leq \binom{4}{4} \): True (both sides are equal to 1).

**Inductive step:** assume \( \binom{3}{3} + \binom{4}{3} + \cdots + \binom{k}{3} = \binom{k+1}{4} \)
for some \( k \geq 3 \).

**Need to show:** \( \binom{3}{3} + \binom{4}{3} + \cdots + \binom{k}{3} + \binom{k+1}{3} \leq \binom{k+2}{4} \)

**Enough to show:** \( \binom{k+1}{4} + \binom{k+1}{3} \leq \binom{k+2}{4} \)

**True by Pascal’s property!**

8. Let \( a_0 = 2 \) and, for \( n > 0 \), let \( a_n = 3a_{n-1} + 4 \). Find and prove an explicit formula for \( a_n \).

Look for a constant \( c \) such that the sequence \( (b_n = a_n + c) \) satisfies \( b_n = 3b_{n-1} \iff (a_n + c) = 3(a_{n-1} + c) \iff a_n = 3a_{n-1} + 2c \). So \( c = 2 \) works.

So: \( a_n + 2 = b_n = 3b_{n-1} = 3^2b_{n-2} = \cdots = 3^n b_0 = 3^n (a_0 + 2) = 3^n \cdot 4 \).

**Answer:** \( a_n = 4 \cdot 3^n - 2 \)
9. Give an example of a bijection $f : \mathbb{Z} \times \{0, 1\} \to \mathbb{Z}$ (recall that $\mathbb{Z}$ is the set of all integers). Find the inverse of your bijection.

\[
\begin{align*}
f(0, 0) &= 0, \quad f(1, 0) = 2, \quad f(2, 0) = 4, \ldots, \quad f(-1, 0) = -2, \quad f(-2, 0) = -4, \ldots \\
f(0, 1) &= 1, \quad f(1, 1) = 3, \quad f(2, 1) = 5, \ldots, \quad f(-1, 1) = -1, \quad f(-2, 1) = -3, \ldots \\
\end{align*}
\]
So $f(n, 0) = 2n$, $f(n, 1) = 2n + 1$.

So $f(n, a) = 2n + a$, ($n \in \mathbb{Z}$, $a \in \{0, 1\}$).

**Inverse function $g : \mathbb{Z} \to \mathbb{Z} \times \{0, 1\}$**

\[
g(b) = (b \div 2, b \mod 2).
\]

10. (a) Find the reciprocal of 10 in $\mathbb{Z}_{63}$ with the help of Euclid’s algorithm.

\[
\begin{align*}
63 &= 6 \cdot 10 + 3 \\
10 &= 3 \cdot 3 + 1 \\
\end{align*}
\]

So $1 = 10 - 3 \cdot 3$.

So $10 = 10 - 3 \cdot (63 - 6 \cdot 10)$.

$= 19 \cdot 10 - 3 \cdot 63$. 

\[\text{So } 19 \cdot 10 \mod 63 = 1 \uparrow \]

\[19 \otimes 10 = 1 \text{ in } \mathbb{Z}_{63}\]

Answer: the reciprocal of 10 in $\mathbb{Z}_{63}$ is 19.

(b) Using your answer to (a), solve the equation $10 \otimes x \otimes 5 = 13$ in $\mathbb{Z}_{63}$.

\[10 \otimes x = 8 \iff x = 8 \otimes 10 = 8 \otimes 19 = 8 \cdot 19 \mod 63 \]

\[= 152 \mod 63 = 26\]