Braid Groups

Intuitive Definition. For $n \geq 2$ let $B_n$ be the group of "braids on $n$ strings" where a braid on $n$ strings is a set of $n$ strings connected two ordered sets of $n$ points and recording at each intersection of the strings whether string $k$ goes over or under string $j$. This is a monoid under concatenation of braids (tacking braids end to end). This monoid has identity: the braid the connects each point two its pair with no crossings and it can be seen that taking a braid and replacing each crossing with the reverse crossing give an inverse.

Algebraic Definition. Now, let $B_n = < T_1, \ldots, T_{n-1} > /$some relations where $T_i$ is the identity braid but with a string connecting the point $i$ to $i+1$ crossing over a line connecting point $i+1$ to point $i$. It is clear that these generate the following relations:

(R1) $T_i T_j = T_j T_i$ for $|i - j| \geq 2$
(R2) $T_i T_{i+1} = T_{i+1} T_i$

By a Theorem due to Artin, $< T_1, \ldots, T_{n-1} > / (R1)(R2)$ is a presentation of $B_n$.

Topological Inclination. To look at the $B_n$'s topologically, we'd really like to find spaces $X_n$ such that $B_n = \pi_1(X_n)$. To this end let $C_n$ be the configuration space of $n$ ordered points $z_1, \ldots, z_n \in \mathbb{C}$ so $C_n = \{ (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j, \forall i \neq j \} = \mathbb{C}^n \setminus \{ z_1 = z_2 \}$ where the right side is $\mathbb{C}^n$ with the hyperplanes $z_1 = z_2$ removed. Now, the symmetric group $S_n$ acts freely on $C_n$ so let $X_n = C_n / S_n$ be the configuration space of $n$ unordered points in $\mathbb{C}$, then by a theorem of Fox-Neuworth $B_n \cong \pi_1(X_n)$. The proof sketch is highly visual and will be omitted.

Generalized Braid Groups

Algebraic Definition. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Then we define the Algebraic Braid Group to be the Weyl group of $\mathfrak{g}$ with the relations $s_i^2$ removed from the presentation:

$$B_{\mathfrak{g}}^{\text{Alg}} = < s_1, \ldots, s_r > / s_is_j\ldots = s_js_i\ldots \forall i \neq j$$

where $r$ is the number of points in the Dynkin diagram of $\mathfrak{g}$ and for each $i, j$ the string on both side of $s_is_j\ldots = s_js_i\ldots$ has length equal to the number associated to the connection between the points $i$ and $j$ in the standard way.

For example let $\mathfrak{g} = \mathfrak{sl}_5$. The Dynkin Diagram is $\circ \xrightarrow{4} \circ$ and so $B_{\mathfrak{sl}_5}^{\text{Alg}} = < s_1, s_2 > / s_1s_2s_1s_2 = s_2s_1s_2s_1$. 


Topological Inclination Revisited. Let \( \mathfrak{h} \) denote the Cartan Subalgebra of \( \mathfrak{g} \). Now, \( \mathfrak{h} \) acts on \( \mathfrak{g} \) by adjoint action and so \( \mathfrak{g} = \mathfrak{h} \oplus \bigoplus \alpha \mathfrak{g}_\alpha \) where \( \alpha \) are the joint eigenvalues of \( \mathfrak{h} \). Then \( C_n \) above corresponds to \( \mathfrak{h} \backslash \bigcup \alpha \ker \alpha \). Then let \( B_{\mathfrak{g}}^{\text{Top}} := \pi_1(\mathfrak{h}_{\text{reg}}/W) \) where \( W \) is the Weyl group of \( \mathfrak{g} \). By a theorem of Brieskorn \( B_{\mathfrak{g}}^{\text{Top}} \cong B_{\mathfrak{g}}^{\text{Alg}} := B_{\mathfrak{g}} \).

Representations of Braid Groups

Topological Representations. The idea here is to construct representations of \( B_{\mathfrak{g}} \) from the monodromy/analytic continuation of solutions of systems of \( n \)'th order ODE's on \( \mathfrak{h}_{\text{reg}}/W \). We have the following general solution:

Let \( X \) be a complex manifold, and let's look at \( \frac{\partial f}{\partial z_i} = A_i f \) where \( f : X \rightarrow \mathbb{C}^n \) and \( A_i : X \rightarrow \text{End}(\mathbb{C}^n) \). Let \( \Phi \) be a solution near \( x_0 = \gamma(0) \), then \( \Phi_\gamma \) is the analytic continuation of \( \Phi \) along \( \gamma \) so \( \Phi : \gamma \rightarrow \text{Gl}_n(\mathbb{C}) \). Let \( \mu(\gamma) = \Phi^{-1}(1)\Phi_\gamma(0) \in \text{Gl}_n(\mathbb{C}) \).

**Proposition.** If the above is integrable eg. \( [\partial_i - A_i, \partial_j - A_j] = 0 \equiv \partial_i A_j - \partial_j A_i = [A_i, A_j] \) then \( \mu(\gamma) \) only depends on the homotopy class of \( \gamma \).

Now, let \( X = \mathfrak{h}_{\text{reg}}, \mathbb{C}^n \rightarrow V \in \text{Rep}(\mathfrak{g}) \). Then what is \( A_i \)? Invariantly we have

\[
\nabla = d - \sum \frac{d\alpha}{\alpha} r_\alpha
\]

where \( r_\alpha \in \text{End}(V) \).

For example let \( a_1, \ldots, a_m \) be a basis of \( \mathfrak{h}^* \) and let \( a^1, \ldots, a^m \) be the dual basis of \( \mathfrak{h} \). Then \( f : \mathfrak{h}_{\text{reg}} \rightarrow V \) and

\[
\frac{\partial f(a)}{\partial a_i} = \left( \sum \frac{\alpha(a^i)}{\alpha(a)} \cdot r_\alpha \right) f
\]

To each \( \alpha \) we associate a 3-dimensional subalgebra \( \mathfrak{sl}_2^\alpha \) of \( \mathfrak{g} \), eg: if \( \mathfrak{g} = \mathfrak{sl}_n \) then \( \alpha = z_i - z_j \), \( \mathfrak{sl}_2^\alpha = \langle E_{ij}, E_{ji}, E_{ij} - E_{ji} \rangle \) and \( C_\alpha = (e_\alpha f_\alpha + f_\alpha e_\alpha + 1/2h^2\alpha)\alpha/2 \). Thus

\[
\nabla_C = d - \sum \frac{d\alpha}{\alpha} C_\alpha
\]

is integrable \( \forall h \in \mathbb{C} \).

**Corollary.** \( \mu_h : \pi_1(\mathfrak{h}_{\text{reg}}/W) \rightarrow \text{Gl}(V) \). These are the topological representations of \( B_{\mathfrak{g}} \).

Algebraic Representations. Algebraic representations of \( B_{\mathfrak{g}} \) come from the quantum group \( \mathcal{U}_h \mathfrak{g} \) which is an algebra depending on a parameter \( h \) and is a deformation of \( \mathcal{U} \mathfrak{g} \). For example, \( \mathfrak{sl}_2 = \langle e, f, h \rangle \) modulo the relations \([h, e] = 2e, [e, f] = h, [h, f] = -2f \)

where

\[
e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]
so quantum $\text{Sl}_2 = \mathcal{U}_h \text{Sl}_2 = < E, F, H >$ modulo the relations $[H, E] = 2E$, $[H, F] = -2F$
and
$[E, F] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}} = H + \sigma(h^2)$

**Theorem.** If $V$ is a representation of $\mathcal{U}_h \mathfrak{g}$ there exists a Weyl group operator $s^h_1 \ldots s^h_r \in \text{GL}(V)$ such that $s^h_i s^h_j = s^h_j s^h_i$ for $\forall i \neq j$. As a corollary there exists a map $\lambda : \hat{B}_\mathfrak{g}^{\text{Alg}} \rightarrow \text{GL}(V)$.

**The Big Theorem.** The two representations of $B_\mathfrak{g}$ described above are the same.

**Affine Setting:**

**Affine Braid Groups.** Let an algebraic affine braid group be define to be $\hat{B}_\mathfrak{g}^{\text{Alg}} := < s_0, \ldots, s_r > / \text{relations depending on the affine Dynkin diagram of } \mathfrak{g}$ in the same way as for non affine braid groups, with $s_0$ being the added affine vertex. Similarly define a topological affine braid group as follows:

Let $G$ be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $H \subset G$ be the maximal torus such that $\text{Lie}(H) = \mathfrak{h}$. Then $H_{\text{reg}} = H \setminus \bigcup \{ e^\alpha = 1 \}$. Then we define $\hat{B}_\mathfrak{g}^{\text{Top}} = \pi_1(H_{\text{reg}}/W)$ and note that similar to the non affine case, $\hat{B}_\mathfrak{g}^{\text{Alg}} = \hat{B}_\mathfrak{g}^{\text{Top}}$.

**Algebraic Representations.** Let $L_\mathfrak{g} = \mathfrak{g}[ht^{-1}]$ an affine algebra. Then, similar to before, $\mathcal{U}_h(L_\mathfrak{g})$ associates to a map $\hat{B}_\mathfrak{g}^{\text{Alg}} \rightarrow \text{GL}(V)$ for $V \in \text{Rep}(\mathcal{U}_h(L_\mathfrak{g}))$

**Topological Representations.** Similar to before we have a connection

$$\nabla = d - h \sum \frac{d\alpha}{e^\alpha - 1} C_\alpha - d\mu_i A^i$$

where the tail of this equation has values in $Y(\mathfrak{g})$, the Yangians of $\mathfrak{g}$.

**Theorem.** There exists a $\nabla$ with coefficients in $Y(\mathfrak{g})$ which is flat.

**Conjecture.** These two representation of affine braid groups are in fact the same.