Semi-Invariants of Tubular Algebras

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Basic Definitions

Throughout this presentation \( Q = (Q_0, Q_1) \) will be a quiver without oriented cycles, where \( Q_0 \) is the finite set of vertices and \( Q_1 \) is the finite set of arrows. If \( a \in Q_1 \) is an arrow then \( ta \) and \( ha \) denote its tail and head respectively.

A path is a sequence of arrows \( p = a_1a_2 \ldots a_s \) with \( ta_i = ha_{i+1} \) for all \( i \). We define \( tp = ta_s \) and \( hp = ha_1 \). For each vertex \( x \in Q_0 \) we also define the trivial path \( e_x \) of length 0, satisfying \( te_x = he_x = x \). An oriented cycle is a nontrivial path satisfying \( hp = tp \).
Let $K$ be an algebraically closed field. The path algebra $KQ$ is the $K$-vector space spanned by all paths (including the paths $e_x$). If $p$ and $q$ are paths, then their product $p \cdot q$ is the concatenation of the paths if $tp = hq$, and is defined 0 otherwise.

The category $\text{Rep}_K(Q)$ of representations of the quiver $Q$ is the category of finite dimensional $KQ$-modules. If $V$ is a representation of $Q$ (i.e., a finite dimensional $KQ$-module) then we define $V(x) = e_x V$ for all $x \in Q_0$ and $V(p) : V(tp) \rightarrow V(hp)$ is the restriction of multiplication with $p$ to $V(tp) = e_{tp} V$ for every path $p$. 
The path algebra is graded

\[ KQ = \bigoplus_{x,y \in Q_0} e_x KQ e_y. \]

Let \( r \in KQ \) be a relation, i.e.,

\[ r = \sum_{i=1}^{s} c_i p_i \]

with \( p_i \) a path and \( c_i \in K \) for all \( i \).
We say that the relation $r$ is \textit{admissible} if $r$ is homogeneous with respect to the grading, i.e., there exist $tr, hr \in Q_0$ such that $tp_i = tr$ and $hp_i = hr$ for all $i$. Let us assume that $I$ is an admissible ideal, i.e., a two sided ideal generated by admissible relations.
We will call $Q/I$ a quiver with relations. The category $\text{Rep}_K(Q/I)$ of representations of $Q/I$ is the category of finite dimensional $KQ/I$-modules. We may assume that $I$ is generated by admissible relations of length $\geq 2$, because otherwise the algebra $KQ/I$ is a factor of a path algebra of a smaller quiver.

A dimension vector for $Q$ is an element $\alpha \in \mathbb{N}^{|Q_0|}$, where $\mathbb{N} = \{0, 1, 2, \ldots \}$ is the set of nonnegative integers. We say that a representation $V$ is $\alpha$-dimensional if $\dim V(x) = \alpha(x)$ for all $x \in Q_0$. 
For a dimension vector $\alpha$ we define the representation space by

$$Rep_K(Q, \alpha) = \bigoplus_{a \in Q_1} \text{Hom}(K^{\alpha(ta)}, K^{\alpha(ha)}).$$

Note that every element

$$V = \{V(a) \mid a \in Q_1\} \in Rep_K(Q, \alpha)$$

can be viewed as a representation of $Q$.

The groups $GL(Q, \alpha) := \prod_{x \in Q_0} GL(\alpha(x))$ and $SL(Q, \alpha) := \prod_{x \in Q_0} SL(\alpha(x))$ act on $Rep_K(Q, \alpha)$ in a natural way.
Two representations $V, W \in \text{Rep}_K(Q, \alpha)$ are isomorphic if they lie in the same $GL(Q, \alpha)$-orbit.

We also define

$$\text{Rep}_K(Q/I, \alpha) \subseteq \text{Rep}_K(Q, \alpha)$$

as the Zariski-closed subset defined by

$\text{Rep}_K(Q/I, \alpha) = \{V \in \text{Rep}_K(Q, \alpha) \mid V(r) = 0 \ \forall r \in I \ \text{homogeneous}\}$. 
The space $\text{Rep}_K(Q/I, \alpha)$ does not have to be irreducible. We denote its irreducible components by $\text{Rep}_K(Q/I, \alpha)_i \ (i = 1, 2, \ldots, N(Q/I; \alpha))$. We are interested in the rings of semi-invariants

$$SI(\text{Rep}(Q/I, \alpha)_i) := K[\text{Rep}(Q/I, \alpha)_i]^{SL(Q, \alpha)}.$$
We recall that the Euler form for $Q$ is a bilinear form on the space $\Gamma := \mathbb{Z}^{Q_0}$ defined by

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$

Every representation $V \in \text{Rep}_K(Q)$ has a canonical resolution

$$0 \to \bigoplus_{a \in Q_1} V(ta) \otimes P_{ha} \to \bigoplus_{x \in Q_0} V(x) \otimes P_x \to V \to 0 \quad (1)$$

where $P_x = KQe_x$ is the indecomposable projective module associated to the vertex $x \in Q_0$. 
More precisely, denoting by $[x, y] := e_y KQ e_x$ the $K$-span of all paths from $x$ to $y$, we have $P_x(y) = [x, y]$ with the linear map $P_x(a)$ acting by the left composition with $a$. We can also characterize $P_x$ by the property $\text{Hom}_R(P_x, W) = W(x)$ for all $W \in \text{Rep}_K(Q)$.

Now (1) implies that $\text{Rep}_K(Q)$ is hereditary, and for $W \in \text{Rep}_K(Q, \beta)$ we have that the Euler characteristic is equal to

$$\chi(V, W) := \dim_K \text{Hom}_R(V, W) - \dim_K \text{Ext}_R(V, W) = \langle \alpha, \beta \rangle.$$
For a quiver $Q$ with relations $I$ we notice that the indecomposable projective modules again correspond to vertices from $Q_0$ and the module corresponding to $x \in Q_0$ is just $P'_x := P_x/IP_x$. They are characterized by the property that for each $W' \in \text{Rep}_K(Q/I)$ we have $\text{Hom}_{R/I}(P'_x, W') = W'(x)$. 
Let \( V' \in \text{Rep}_K(Q/I, \alpha) \). We construct the module
\[
\bar{P}_0 = \bigoplus_{x \in Q_0} V'(x) \otimes P'_x.
\]
Then for each \( V' \in \text{Rep}_K(Q/I, \alpha) \) the kernel
\[
0 \rightarrow V'_{(1)} \rightarrow \bar{P}_0 \rightarrow V' \rightarrow 0
\]
has the same dimension vector. We define \( \bar{P}_1 \) by using the construction of \( \bar{P}_0 \) for \( V'_{(1)} \). Continuing like that we construct the family of projective resolutions of modules from \( \text{Rep}_K(Q/I, \alpha) \) with fixed terms.
This construction allows to define the Euler form for the quiver $Q$ with relations $I$. For two dimension vectors $\alpha$ and $\beta$ we set

$$\langle\langle \alpha, \beta \rangle \rangle = \sum_{s \geq 0} (-1)^s \dim_K \Ext^s_{R/I}(V', W')$$

where $V'$, $W'$ are the modules from $\alpha, \beta$ respectively.
Assume $Q/I$ is a quiver with relations, $\alpha$ a dimension vector. Every representation $V$ of dimension vector $\alpha$ of projective dimension 1 with a (minimal) projective resolution

$$0 \to P_1 \to P_0 \to V \to 0$$

defines a determinantal semi-invariant $c^V$

$$W \mapsto \det(\text{Hom}_{Q/I}(P_0, W) \to \text{Hom}_{Q/I}(P!, W))$$

on all components of $\text{Rep}(Q/I, \beta)$ such that $\langle \langle \alpha, \beta \rangle \rangle = 0$. Such semi-invariant might of course be identically zero on some components.
A component $C$ of $\text{Rep}(Q/I, \beta)$ is faithful if the ideal $J = \{x \in KQ \mid x|_C = 0\}$ is equal to $I$, i.e. there are no extra relations satisfied on $C$.

**Theorem** (Derksen and Weyman) Let $Q/I$ be a quiver with relations. Let $C$ be a faithful component of $\text{Rep}(Q/I, \beta)$. Then the ring of semi-invariants $SI(Q/I, C) := SI(\text{GL}(Q, \beta), K[C])$ is spanned by the determinantal semi-invariants $c^V$. 
**Tubes and the rings of semi-invariants**

Let $\Lambda = KQ/I$ and denote $\text{Rep}(\Lambda, \beta)$ the algebraic variety of the $\Lambda$-modules of dimension vector $\beta$.

The family of tubes over $\Lambda$ of ranks $(m_1, \ldots, m_s)$ is the set of data given below. First, we have a family of indecomposable modules $\{V_t\}$ ($t \in \mathbb{P}^1$) of dimension $h$, and the set of modules $E_{i}^{(j)}$ of dimension vectors $e_{i}^{(j)}$ ($1 \leq j \leq s$, $0 \leq i \leq m_j$), all of projective dimension $1$. The smallest category $\text{Reg}(\Lambda)$ containing modules $V_t$ and $E_{i}^{(j)}$, closed under the extensions and direct summands, is called the category of regular $\Lambda$-modules.
A vector

\[ \beta = ph + \sum_{j=1}^{s} \sum_{i=0}^{m_j-1} p_{ij} (j) e_{ij} (j) \]

(where we assume that for each \( j = 1, \ldots, s \) we have
\( \min \{ p_{ij} (j) ; 1 \leq i \leq m_j - 1 \} = 0 \)) is called a regular
dimension vector.

We require that for every regular dimension vector \( \beta \)
there exists an irreducible component \( \text{Reg}(\Lambda, \beta) \) of
\( \text{Rep}(\Lambda, \beta) \)
Reg(Λ, β) is subject to the following conditions:

1. The dimension vectors $e^{(j)}_i$ satisfy the relations

$$h = \sum_{i=0}^{m_j-1} e^{(j)}_i.$$

for $j = 1, \ldots, s$, and the dimension of the space spanned by $e^{(j)}_i$ equals $\sum_{j=1}^{s} m_j - s + 1$,

2. The decomposition of a general vector in Reg(Λ, β) is given by the same formula as for the extended Dynkin quivers
3. Dimension vectors $e_i^{(j)}$, and the vectors
\[ e_{i,k}^{(j)} := e_i^{(j)} + e_{i+1}^{(j)} + \ldots + e_k^{(j)} \]
are Schur roots and the generic modules $E_{i,k}^{(j)}$ have projective dimension 1 and injective dimension 1 over $\Lambda$

4. The values of the Euler form $\langle \langle e_i^{(j)}, e_k^{(l)} \rangle \rangle = 0$ if $j \neq l$, and $\langle \langle e_i^{(j)}, e_i^{(j)} \rangle \rangle = -1$, $\langle \langle e_i^{(j)}, e_{i+1}^{(j)} \rangle \rangle = -1$, $\langle \langle e_i^{(j)}, e_i^{(j)} \rangle \rangle = 1$
5. The general module in dimension vector $Reg(\Lambda, h)$ is a 1-parameter family of modules $V_t \ (t \in K \cup \infty)$, which are also of projective dimension 1 and of injective dimension 1.

6. Every indecomposable module $X$ of projective dimension $\leq 1$ orthogonal to $Reg(h)$ (in the sense that $Hom_\Lambda(X, V_t) = Ext^1_\Lambda(X, V_t) = 0$ for general $t$) is in the category $Reg(\Lambda)$.

7. The condition 6. implies by results of [DW6] the existence of the semi-invariant $c^{Vu}$ in the coordinate ring $SI(\Lambda, \beta)$. 
Theorem. Let \( \{V_t, E_i^{(j)}\} \) be a family of tubes of ranks \((m_1, \ldots, m_s)\). Let \( \beta \) be a regular dimension vector. Then the ring of regular semi-invariants \( SI_{reg}(\Lambda, \beta) \) has the generators and relations described as follows.

a) The non-homogeneous generators are \( c^{E_{[i,k]}^{(j)}} \) where \([i, k]\) is an "admissible" path on the \( j \)-th circle,

b) The homogeneous generators \( c^{V_t} \) which span the space of dimension \( p + 1 \). We fix a basis \( \{c_0, \ldots, c_p\} \) of the weight space \( SI_{reg}(\Lambda, \beta) \langle h, - \rangle \).
c) There are $s$ relations in $SI_{reg}(\Lambda, \beta)$ each expressing the product of non-homogeneous semi-invariants of index zero on the $j$-th circle as a linear combination of the semi-invariants $c_0, \ldots, c_p$. 
Tubular Algebras

Give a finite dimensional algebra $A_0$ and an $A_0$ module $R$ we denote by $A_0[R]$ the one-point extension of $A_0$ by $R$ namely the algebra

$$
\begin{bmatrix}
A_0 & R \\
0 & k
\end{bmatrix}
$$

$$
= \left\{ \begin{bmatrix} a & r \\ 0 & b \end{bmatrix} \mid a \in A_0, r \in R, b \in k \right\}
$$
The quiver of $A_0[R]$ contains the quiver of $A_0$ as a full subquiver and there is an additional vertex $\omega$ called the extension vertex of $A_0[R]$.

Similarly, the one-point co-extension $[V]A$ of $A$ by $V$ is defined by $[V]A = ((A^{op})[DV])^{op}$
Let $A_0$ be an algebra $E_1, \cdots, E_t$ be $A_0$ modules and $K_1, \ldots K_t$ branches. Let $A = A_0[E_i, K_i]_{i=1}^t$ be inductively defined.

The algebra $A$ is called a tubular extension of $A_0$ using modules from the tubes provided that the modules $E_1, \ldots E_t$ are pairwise orthogonal modules from the mouth of the tubes.
Algebras of the form \( \text{End}(T) \) where \( T \) is a preprojective tilting module of a tame hereditary algebra are called **tame concealed algebras**.

A tubular extension of a tame concealed algebra of tubular type \((2,2,2,2), (3,3,3), (4,4,2)\) or \((6,3,2)\) is called a **tubular algebra**

An algebra will be said to be cotubular provided that the opposite algebra \( A^{op} \) is tubular.
Let $A = (A_0, A_\infty)$ be an algebra which is an extension of $A_0$ and a coextension of $A_\infty$.

Let $\alpha_0$ be the positive radical generator of $A_0$ and $\alpha_\infty$ be the positive radical generator for $A_\infty$. Then $h = p\alpha_0 + q\alpha_\infty$ with $p, q \in \mathbb{Z}_+$.

For the one parameter family of modules $V_t$, we have that $t = \frac{q}{p}$. 
Shrinking Functors

For a tubular algebra \( A = (A_0, A_\infty) \), we can define a left shrinking functor \( \Sigma_L = \text{Hom}(T, -) \) with \( T = T_0 + T_P \) where \( T_0 \) is in the preprojective component of \( A \) and \( T_P \) is a projective module not in the preprojective component.

Similarly we can define a right shrinking functor \( \Sigma_R = \text{Hom}(-, S) \) where \( S = S_0 + S_Q \) with \( S_0 \) in the preinjective component and \( S_Q \) a injective module not in the preinjective component.

We can associate a linear transformation with \( \Sigma_L \) and \( \Sigma_R \) namely \( \sigma_l \) and \( \sigma_r \).
There exists left and right shrinking functors such that

\[(p\alpha_0 + q\alpha_\infty)\sigma_l = (p + q)\alpha_0 + q\alpha_\infty\]

and

\[(p\alpha_0 + q\alpha_\infty)\sigma_r = p\alpha_0 + (p + q)\alpha_\infty\]

So basically we can use shrinking modules to shift from 
\(t = 1\) to \(t = \frac{q}{p}\) for any \(h = p\alpha_0 + q\alpha_\infty\).