Introduction to abstract polytopes

Mark Mixer

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In the theory of convex polytopes, the "basic objects" are the polytopes themselves, and we define the faces in terms of supporting hyperplanes. When we move to the combinatorial realm, we want to instead start with the faces and use them to build up polytopes; we don't have the benefit of Euclidean space or hyperplanes!

An abstract polytope will be defined as a partially ordered set of faces, such that it satisfies 4 properties. To fully state the properties, we need some notation. In the definitions that follow, let P be a poset of faces. We say that faces F and G are *incident* if $F \leq G$ or $G \leq F$. A *chain* of P is a totally ordered subset of P. A *flag* of P is a maximal chain. We use $\mathcal{F}(P)$ to denote the set of flags of P.

Whenever we have faces $F \leq G$, we can form the *section* $G/F := \{H \in P \mid F \leq H \leq G\}$. We say that a section is *connected* if for all $H, H' \in G/F$, there is a finite sequence of proper faces (faces other than F and G):

$$H = H_0, H_1, \ldots, H_{k-1}, H_k = H'$$

such that H_i and H_{i+1} are incident, for $i = 0, \ldots, k-1$. We say that P is *strongly connected* if all of its sections are connected.

Now we are ready to give the full definition. An *abstract polytope of rank n* (hereafter referred to simply as a "polytope (of rank n)"), is a poset P satisfying:

(P1) P has a greatest face F_n and a least face F_{-1}.

(P2) Every flag of P has $n+2$ faces (including F_n and F_{-1}). Note that this induces a natural rank function on P.

(P3) P is strongly connected.

(P4) For any $i = 0, 1, \ldots, n-1$, if $F \leq H \in P$ are incident faces such that $\text{rank}(F) = i-1$ and $\text{rank}(H) = i+1$, then there are exactly two faces G_1 and G_2 such that $F < G_j < H$ for $j = 1, 2$. (This is informally called the "diamond property").

In order for this to be a good definition, we should have that all convex polytopes naturally give rise to abstract polytopes. So let us justify the properties above by looking at what they mean for convex polytopes.

(P1) The empty set serves as F_{-1} for a convex polytope, and the whole polytope serves as F_n.

(P2) It's clear that the flags of a convex polytope contain elements of all dimensions from -1 to n; this is sort of like saying that no face of our polytope is hollow.

(P3) We start by noting that in order for a set to be convex, it must be path-connected, so a convex polytope is already connected. Furthermore, in order for its facets and vertex figures to be polytopes, they must be connected as well. By repeatedly taking facets and vertex figures, we see that this means that all sections must be connected.

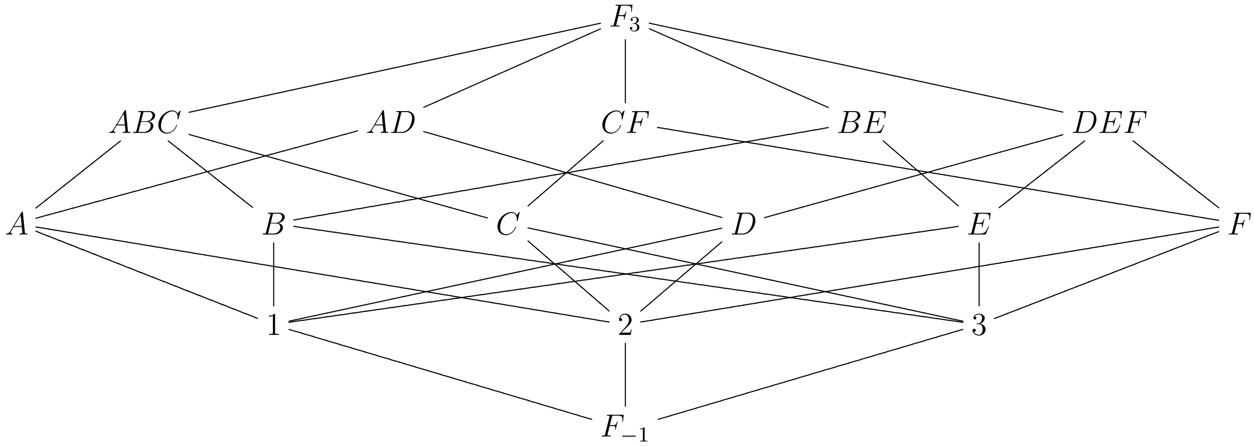
(P4) This condition is also true for convex polytopes: edges contain two vertices; a given vertex on a 2-face belongs to exactly two edges of that face; and so on.

Finally, there is a condition that is equivalent to P3 for any poset satisfying P1 and P2, and it is sometimes more convenient to work with: we say that P is strongly flag-connected if each section G/F of P is flag-connected, in the sense that given any two flags Φ and Ψ of G/F, there is a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that Φ_i and Φ_{i+1} differ only in a single face, for $i = 0, \ldots, k-1$.

The above explanations show that the face-lattice of a convex polytope is indeed an abstract polytope. However, there are many more abstract polytopes. Let us look at some examples.

Example: By starting with the cube and identifying opposite vertices (i.e., vertices separated by the internal diagonal of the cube), we get the *hemicube*, which can't be imbedded in Euclidean space. The face-lattice of the hemicube is an abstract polytope.

Example: For convex polytopes, the face-lattice is indeed a lattice; that is, any two faces have a unique join (least upper bound) and a unique meet (greatest lower bound). However, an abstract polytope doesn't have to be a lattice. Suppose we start with two triangles; one has edges A, B, C and the other has edges D, E, F. Bend the triangles and glue them together at their vertices such that the edges are separated from each other. To each of these three separations, we can glue a digon. Then you get the following abstract polytope:



This is not a lattice: the edges A and D have two greatest lower bounds; namely, the vertices 1 and 2.

Example: Suppose we start with an infinite tessellation of the plane by squares meeting 4 at each vertex. Now let us identify two vertices if they are separated by 3 units in the horizontal or the vertical direction. These identifications cause the tessellation to collapse to a finite 3x3 grid of squares, with the 4 corners identified and with opposite edges identified. Then the face-set of this is an abstract polytope; in fact, this corresponds to a finite tessellation of the torus.

Having generalized the notion of polytopes, we'd also like to generalize the notion of regularity. For convex polytopes, there are several equivalent definitions. One of the most appealing, due to its simplicity, is that we define regular polygons in the usual way (i.e., equilateral and equiangular), and then say that a convex n-polytope is regular whenever its facets are all isomorphic and are all regular. However, when we try to generalize this inductive definition for abstract polytopes, we run into a problem. Whenever an abstract polytope is the face-lattice of a regular convex polytope, we want the abstract polytope to be regular as well. Since there are regular p-gons for any $p \geq 3$, and since the face-lattice of a regular p-gon is the same as that of an p-gon that is not regular, we must have that all abstract p-gons are regular. This isn't a problem, but now we conclude that the regular abstract polyhedra would simply be polyhedra where all the facets are p-gons for some fixed p. And this is a very permissive definition; it's possible to build abstract polyhedra that are "regular" in this sense, but that have a trivial symmetry group!

Of course, we need to say what we mean by symmetries in this setting. In the convex setting, the symmetries we cared about were isometries that carried the polytope to itself. Now we no longer care about any ambient geometry, and our notion of symmetry is much broader. Given an abstract polytope P, a function $\varphi : P \to P$ is a *symmetry* or a *combinatorial automorphism* if φ and φ^{-1} are rank and incidence preserving (which additionally forces φ

to be a bijection on the faces). We define $\Gamma(P)$ to be the automorphism group of P.

With this group in hand, we can now better generalize regularity. One possible definition would be that P is regular if $\Gamma(P)$ acts transitively on the set of i-faces for each i. However, there is an even stronger definition, and this is the one that is used: *a polytope P is regular if $\Gamma(P)$ acts transitively on the flags of P.*

Now let us briefly state some properties of regular polytopes. In the following, let P be a regular polytope, and let $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$ be a flag of P that we'll call the *base flag.*

Proposition: If P is finite, then the number of flags of P is equal to $|\Gamma(P)|$.

Proposition: For $i = 0, 1, \ldots, n-1$, there is an involution $\rho_i \in \Gamma(P)$ such that $\rho_i(\Phi)$ is the unique flag that is i-adjacent to Φ; that is, that differs from Φ only in the i-face. (Existence of such a flag follows from P4, and existence of such an automorphism follows from regularity).

Proposition: If $|j - k| \geq 2$, then $(\rho_j \rho_k)^2 = 1$ (i.e., $\rho_j \rho_k = \rho_k \rho_j$.)

Proposition: $\Gamma(P)$ is generated by the involutions $\rho_0, \ldots, \rho_{n-1}$. In particular, this means that $\Gamma(P)$ is a quotient of a string Coxeter group with n generators.

Finally, we close with a few remarks about graphs related to polytopes. Given a polytope P of rank $2d$, its *medial layer graph* $M(P)$ is the subgraph of the Hasse diagram of P induced by faces of rank d and $d-1$. Then we can ask: what properties does $M(P)$ have? For example, what is the arc-transitivity of this graph. Conversely, if we start with a graph G, when is it the medial layer graph of some polytope P? Understanding how graph-theoretic properties correspond to properties of the polytope can help us better understand what polytopes exist and how they are structured.