Central simple modules of a Gorenstein Artin algebra

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Abstract: Let $R$ be the polynomial ring in $r$ variables over a field $k$. Let $A = R/I$ be a standard Artinian algebra quotient of $R$, not necessarily graded. For $A$ graded and a given linear form $z$ in $A_1$, T Harima and J. Watanabe in a series of papers studied the “central simple modules” (CSM) of the pair $(A, z)$: these are the nonzero factor modules (here $c$ is the socle degree of $A$) of the sequence

$$(0 : z^c) + (z) \supset (0 : z^{c-1}) + (z) \supset \cdots \supset (0 : z) + (z).$$

They related these to the Lefschetz properties of $(A, z)$. The latter have to do with the Jordan block decomposition of the multiplication map $m_z$ ”multiply by $z$” on $A$. In particular Harima and Watanabe showed that the CSM’s have symmetric Hilbert functions when $A$ is Gorenstein graded. We will describe these concepts, give examples, and suggest how they might generalize to non-graded $A$ that are Gorenstein. We also state some open questions.
1 Introduction: Gorenstein Artin algebras

$k =$ alg. closed field. $R = k[x,y]$ or $k[x,y,z]$ polynom, ring.
Let $I=$ ideal primary to the maximal ideal $m = (x,y)$ or $ (x,y,z) :$ so $I \supset m^{c+1}, I \not\supset m^c$ some $c =$ socle degree of $A.$

An Artin algebra $A = R/I, \dim_k A = n < \infty.$

Soc$(A) = (0 : m) = \{a \in A \mid ma = 0\}$

$A$ is Gorenstein Artinian (GA) if $\dim_k \text{Soc}(A) = 1.$ Then

$\text{Soc}A = A_c.$ Let $\phi : A_c \rightarrow k.$ Then $< \cdot, \cdot > : A \times A \rightarrow k,$

$< a, b >_{\phi} = \phi(ab)$ is a non-degenerate bilinear form.

Let $S = k[X,Y]$ or $k[X,Y,Z].$ $R$ acts on $S$ as PDO w.o. coeffs (contraction): $x^i \circ X^u = X^{u-i}$ if $u \geq i,$ or 0 otherwise.

Let $F \in S,$ set Ann $(F) = \{f \in R \mid f \circ F = 0\}.$

Let $I \subset R$ ideal. Set $I^\perp = \{F \in S \mid I \circ F = 0\}$

Thm (F.H.S. Macaulay, 1915). $A$ is Gorenstein Artin $\iff$

$\exists F \in S \mid A = R/I, I = \text{Ann} (F).$

Associated graded alg. $A^* = Gr_m(A) = A_0 \oplus A_1 \oplus \cdots \oplus A_c,$
\[ A_i = m^i A / m^{i+1} \cong (I \cap m^i + m^{i+1}) / (m^{i+1}). \]

Hilbert function \( H(A) = (h_0, h_1, \ldots, h_c), h_i = \dim_k A_i. \)

**Ex 1.** Let \( I = (y^2, x^3) \). Then \( A = \langle 1, x, x^2, y, yx, yx^2 \rangle \),\n\( \text{Soc}(A) \cong A_3 = \langle yx^2 \rangle \). Here \( I = \text{Ann} \langle X^2 Y \rangle \). \( H = (1, 2, 2, 1) \).

**Ex 2.** Let \( I = \text{Ann} \langle X^2 Y + Y^2 \rangle = (y^2 - x^2 y, x^3) \) "

**Ex 3.** Let \( I = \text{Ann} \langle X^4 - Y^2 \rangle = (xy, y^2 + x^4) \).
\( H = (1, 2, 1, 1, 1) \) \( A = \langle 1, x, y, x^2, x^3, x^4 \rangle \).

**Thm.** (F.H.S. Macaulay), \( R = k[x, y] \): A Gorenstein \( \iff \) A CI.
\( (I = (f, g). \) two generators. Analogue not true for \( k[x, y, z]. \)

**Ex 4.** \( F = X^3 + Y^3 + Z^3, I = (xy, xz, yz, x^3 - y^3, x^3 - z^3) \).

Question: How can you tell when you have all the generators of the ideal?

Answer: The dual module \( \hat{A} = R \circ F \) has the same Hilbert function as \( A \). Here
\( R \circ F = 1; X, Y, Z; X^2, Y^2, Z^2 \) \( F \) of Hilbert function \( H(\hat{A}) = (1, 3, 3, 1) = H(A) \).

One checks that \( J = (xy, xz, yz, x^3 - y^3, x^3 - z^3) \in \text{Ann} (F) \). Then, since \( H(R) = (1, 3, 6, 10, \ldots) \), and \( H(R/J) = (1, 3, 3, 1) \) can be seen directly, we have \( J \subset I \) but \( H(R/J) = H(R/I) \Rightarrow I = J \). The calculation is made in algebra programs.

Question: And when \( A \) is not graded? Answer: Go from \( \hat{A} \) to \( \hat{A}^* \) (top degree forms)
and from \( I \) to \( I^* \) (initial - low degree form) and verify \( H(\hat{A}^*) = H(R/J^*) \). See [I]
Thm. (F.H.S. Macaulay). $A$ graded GA $\Rightarrow H(A)$ symmetric and $\langle \cdot, \cdot \rangle_{\phi} : A_i \times A_{j-i} \rightarrow k$ is an exact duality.

Thm. (J. Watanabe [W], also [I, Prop 1.7]) Assume $A$ GA. Then $H(A)$ symmetric $\iff A^*$ Gorenstein.

In Ex 2, $A^*$ same as in Ex. 1, $H(A) = (1, 2, 2, 1)$, symmetric.

In Ex. 3, $A^* \cong R/(y^2, xy, x^5)$, $H(A) = (1, 2, 1, 1, 1)$.

Here $A^*$ is not Gorenstein, as $H$ is not symmetric.

Q1. What symmetry or duality property does $A^*$ inherit?

Ans. $A^*$ has a descending series of ideals whose successive quotients are reflexive modules [I]. (see below).

Let $z \in A, \bar{z} \neq 0 \in A_1$. $m_z : A \rightarrow A$, $m_z(a) = z \cdot a$.

Let $P_z =$ partition of Jordan blocks of $m_z$.

Q2. Given $A$ what $P_z$ are possible? What is $P_z$ for $z$ generic?

Lefschetz property: $(A, z)$ is strong Lefschetz (SL) if $m_z$ has partition $H^\vee$ (dual to $H$). $A$ has SLP if $\exists\ z \mid (A, z)$ is SL.

(See $m_z$ in Ex. 1*, or $m_x$ in Ex. 3* below.)
2 Invariant spaces for $m_z$

**Ex 1**. $A = R/I, I = (y^2, x^3)$. $H = (1,2,2,1)$.

$m_x$: invariant subspaces $\langle 1, x, x^2 \rangle; \langle y, yx, yx^2 \rangle$. $P_x = (3,3)$.

$k[x]$ generators $\langle 1, y \rangle$.

$m_y$: invariant subspaces $\langle 1, y \rangle, \langle x, yx \rangle, \langle x^2, yx^2 \rangle$. $P_y = (2,2,2)$.

$k[y]$ generators $\langle 1, x, x^2 \rangle$.

$m_z, z = x + y$: invt.sp. $\langle 1, z, z^2, z^3 \rangle, \langle x - y, x^2 \rangle$. $P_z = (4,2)$.

$k[z]$ generators $\overline{1}$ and $\overline{x - y}$. $P_z = H^\vee$, so $(A, z)$ is SL.

**Ex 3**. $A = R/I, I = (xy, y^2 + x^4), H=(1,2,1,1,1)$.

$m_y$: invt. spaces $\langle 1, y, y^2 \rangle; \overline{x, x^2, x^3}, P_x = (3,1,1,1)$.

$m_x$: invariant subspaces $\langle 1, x, x^2, x^3, x^4 \rangle; \langle y \rangle$. $P_x = (5,1)$.

**Ex 3** cont.: $P_x = (5,1) = H^\vee$. So $(A, x)$ is SL.

**Thm.** A CI quotient of $R = k[x,y] \Rightarrow A$ is SL.

**Q.** (open) A height three CI or Gorenstein $\Rightarrow A$ is SL?
T. Harima and J. Watanabe consider for \( z \in m - m^2 \) in \( A \),

\[
(0 : z^c) + (z) \supset (0 : z^{c-1}) + (z) \supset \cdots \supset (0 : z) + (z) \quad (**) 
\]

**Def.** A *central simple* module \( U_i(z) \) for \((A, z)\) is a nonzero quotient of successive terms in (**) \(^1\)

\[
U_i(z) = ((0 : z^i) + (z))/((0 : z^{i-1} + (z)) ,
\]

**Ex 1**. \( A = R/(y^2, x^3) \), The \( m_x \) generators are CS module:

\[
U_3(x) = \langle \overline{1}, y \rangle = A/(x) = (0 : x^3) + (x)/(0 : x^2) + (x).
\]

Recall that \( P_x = (3, 3) \). Writing the multiples of \( U_3(x) \) by \( \{1, x, x^2\} \), and (on the right) their Hilbert functions, we find

\[
\begin{align*}
U_3(x) &= \langle \overline{1}, y \rangle & 1 & 1 \\
xU_3(x) &= \langle \overline{x}, xy \rangle & 0 & 1 & 1 \\
x^2U_3(x) &= \langle \overline{x^2}, x^2y \rangle & 0 & 0 & 1 & 1 \\
H(A) &= 1 & 2 & 2 & 1
\end{align*}
\]

\(^1\)We here use a different notation than [HW], where the \( i \)-th nonzero \( U_i \) is corresponds to a part of \( P_z \).
The $m_y$ generators are the CS-module $U_2(y)$

$$U_2(y) = \langle 1, x, x^2 \rangle = A/(y) = (0 : y^2) + (y)/(0 : y) + (y).$$

Recall that $P_y = (2, 2, 2)$. Writing the multiples of $U_2(y)$ by $\{1, y\}$, and their Hilbert functions, we find

$$U_2(y) = \langle 1, x, x^2 \rangle \quad 1 \quad 1 \quad 1$$

$$yU_2(y) = \langle y, xy, x^2y \rangle \quad 0 \quad 1 \quad 1 \quad 1$$

$$H(A) = \quad 1 \quad 2 \quad 2 \quad 1$$

The $m_z, z = x + y$ generators are two CS modules:

$$U_4(z) : \langle 1 \rangle = A/(z^4) + (z) = ((0 : z^4) + (z))/((0 : z^3) + (z)).$$

$$U_2(z) = \langle x - y \rangle = ((0 : z^2) + (z))/((0 : z) + (z)).$$

Writing the corresponding submodules of $A$, we have

$$k[z]U_4(z) = \langle 1, z, z^2, z^3 \rangle \quad 1 \quad 1 \quad 1 \quad 1$$

$$k[z]U_2(z) = \langle x - y, z(x - y) \rangle \quad 0 \quad 1 \quad 1$$

$$H(A) = \quad 1 \quad 2 \quad 2 \quad 1$$

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2.1 Dual sequence $W_i$ to $U_i$

We have, dually, a sequence

$$(0 : z) \supset 0 : z \cap (z) \supset (0 : z) \cap (z^2) \supset \ldots$$

whose successive quotients are

$$W_i(z) = (0 : z) \cap (z^{i-1}) / ((0 : z) \cap (z^i)) .$$

**Thm** (Harima-Watanabe) A graded Artinian Gorenstein $\Rightarrow$

a. Each $U_i(z)$ has symmetric (palindrome) Hilbert function.

b. $W_i(z) = z^{i-1}U_i(z)$.

c. A SL $\iff \exists z \in A_1 \mid$ each $U_i(z)$ is SL.

d. A SL $\iff$ for generic $z \in A_1$, each $U_i(z)$ is concentrated in a single degree.

e. The pairing $\langle \cdot , \cdot \rangle_{\phi}$ defines an exact pairing $U_i \times W_i \to k$.\footnote{This is implicit in [HW], and is generalized in [Bol]}
\textbf{Ex.} $A = R/(y^2, x^3), \quad \text{Soc} \, A = \langle x^2y \rangle. \quad \phi(x^2y) = 1.
U_3(x) = \langle 1, y \rangle, \, W_3(x) = \langle x^2, yx^2 \rangle.$

Then $\langle \cdot, \cdot \rangle_\phi : U_3 \times W_3 \rightarrow k$ is exact. $1 = (yx^2)^\vee, y = x^2)^\vee.$

\textbf{Q.} Why is $U_i(z)$ called a “central simple” module?\textsuperscript{3}

\textbf{Ans.} Let $\mathcal{C}(z) =$ centralizer of $m_z$ in $\text{End}(A) \cong \text{Mat}_n(k)$,
Then $\text{End} \, U_i$ is a simple $\mathcal{C}(z)$ module, and

$$\pi : \mathcal{C}(z) \rightarrow \prod_i \text{End}(U_i)$$

is the canonical map to a semisimple $\mathcal{C}(z)$ module, with kernel the Jacobson radical of $\mathcal{C}(z)$.

\textbf{Ex.} $A = R/(y^2, x^3), P_x = (3, 3). \, M \in \mathcal{C}(x), U_3 = \langle 1, y \rangle$

\begin{align*}
1 & \quad x & x^2 & y & yx & yx^2 \\
1 | & & & & & \quad a & b & c & d & e & f \\
-x | & 0 & a & b & 0 & d & e \\
M = & x^2 | & 0 & 0 & a & 0 & 0 & d & , \quad M \rightarrow \pi(M) = \left( \begin{array}{cc}
a & d \\
g & j \end{array} \right) \in \text{End}(U_3) \\
 & y | & g & h & i & j & k & l \\
 & yx | & 0 & g & h & 0 & j & k \\
 & yx^2 | & 0 & 0 & g & 0 & 0 & j \\
\end{align*}

\textsuperscript{3}Notation introduced by T. Harima and J. Watanabe.
3 Descending sequence of ideals in $A^*$.

We now consider nongraded Gorenstein Artinian $A$.

**Thm. [I]** $Q(a)$ decomposition of $A^*$. Let $A$ be Gor. Artin. Then $A^*$ has a stratification by a descending sequence of ideals

$$A^* = C(0) \supset C(1) \supset \cdots \supset C(c - 1) = 0,$$

(3.1)

whose successive quotients $Q(a) = C(a)/C(a+1)$ are reflexive $A^*$ modules; with ”reflection degree” $(j - a)/2$. Also $Q(0)$ is a graded Gorenstein algebra. The ideal $C(a)$ satisfies,

$$C(a)_i = (m^i \cap (0 : m^{j+1-a-i})) / (m^{i+1} \cap (0 : m^{j+1-a-i})).$$

(3.2)

The pairing $\langle \cdot, \cdot \rangle_\phi$ on $A$ induces an exact pairing

$$\langle \cdot, \cdot \rangle_{\phi,a} : Q(a)_\nu \times Q(a)_{j-a-\nu} \to k,$$

Ex.3** Let $A = R/I, I = \text{Ann } \langle X^4 - Y^2 \rangle = (xy, y^2 + x^4)$.

Recall $A^* \cong R/(y^2, xy, x^5), H(A) = (1, 2, 1, 1, 1)$.

$Q(0) = R/\text{Ann}(X^4) = R/(y, x^5); \quad H(Q(0)) = (1, 1, 1, 1, 1)$.

$Q(2) = \overline{y}; \quad H(Q(2)) = (0, 1, 0)$. 

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**Thm** [BoI] Let $A$ be Gorenstein Artin. Then

(a.) $W_i(z) = z^{i-1}U_i(z)$.

(b.) The pairing $\langle \cdot, \cdot \rangle_\phi$ defines an exact pairing $U_i \times W_i \to k$.

**Q.** Does the palindrome property extend? We think so.

Proof idea: We may replace $m$ by an ideal $J \subset m$ in (3.2) to define $Q^J(a) = C^J(a)/C^J(a + 1)$ over $Gr_J(A)$.

We may also doubly grade by two ideals $J \subset m, K \subset m$, obtaining $Q^{J,K}(a, b)$, subquotients of $Gr_{J,K}(A)$.

**Thm.** [BoI] $Q^J(a)$ and $Q^{J,K}(a, b)$ decomposition. $Q^J(a)$ and $Q^{J,K}(a, b)$ are reflexive $Gr_J(A)$, or $Gr_{J,K}(A)$ modules.

Take $J = L$ and $A$ graded to obtain Harima-Watanabe reflexivity. Or take $J = L$ and $K = m$ to study the Hilbert function $H_m(U_i)$ (grade with respect to powers of $m$).

Let $M$ be a $Gr_J(A)$ module; we denote by $H_J(M)$ the $J$-graded Hilbert function

$$H_J(M)_i = \dim_k (J^i M / J^{i+1} M) = \dim_k Gr_J(M)_i. \quad (3.3)$$
**Cor.** We have that $H_J(Q^J(a))$ is symmetric about $(j - a)/2$.

Furthermore,

$$H_m(A) = \sum_{a=0}^{j-2} H_m(Q^J(a)).$$  \hspace{1cm} (3.2)

**Ex.** Let $F = x^5y + x^2z^3 + xy^3z$, $H = (1, 3, 6, 6, 3, 2, 1)$, $j = 6$.

$I = \text{Ann } F = (z^3 - x^3y, yz^2, y^2z - x^4, xz^2 - y^3, x^2yz, x^2y^2, x^3z)$

$I^* = (z^3, yz^2, y^2z, y^3 - xz^2, x^2yz, x^2y^2, x^3z, x^6)$.

$Q(0) \cong R/\text{Ann } (x^5y) \cong k[x, y]/(y^2, x^6)$; \hspace{1cm} $H(0) = (1, 2, 2, 2, 2, 2, 1)$.

Here $H(1) = (0, 1, 4, 4, 1)$, and $Q(1)$ has representatives,

$$z; z^2, yz, y^2, xz; y^3, xzy, xy^2, x^2z; xy^3.$$  

Here $P_y = (4, 4, 4, 4, 2, 2, 1, 1)$.

The duality between $U_i, W_i$, is the duality between the top and bottom of the reflexive module $Q^{(y)}(j + 1 - i)$.  

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We have $U_4(y) = Q^{(y)}(3)_0$ and $W_4(y) = Q^{(y)}(3)_3$ satisfy

$$U_4 = \left( (0 : y^4) + (y) \right) / \left( (0 : y^3) + (y) \right) = \langle \begin{array}{cc} 1 & x \\ z & xz \end{array} \rangle.$$ 

$$W_4 = (y^3) = \langle \begin{array}{cc} y^3z & xy^3z \\ y^3 & xy^3 \end{array} \rangle = \langle \begin{array}{cc} x^4y & x^5y \\ y^3 & xy^3 \end{array} \rangle.$$ 

We have

$$U_4 \times W_4 \rightarrow k: \quad 1 = (x^5y)^\vee, \; x = (x^4y)^\vee, \; z = (xy^3)^\vee, \; xz = (y^3)^\vee.$$ 

Notice also that $\times(x + z)$ acting on $U_1 = Q^{(y)}(3)_0$ takes

$1 \rightarrow x + z \rightarrow 2xz \rightarrow 0$, and $x - z \rightarrow 0$, so has partition $(3, 1)$, and $x + z$ is a strong Lefschetz element for $U_1$.

We have $Q^{(y)}(5)$ comprised of $U_2 \cong Q^{(y)}(5)_0$ and $W_2 \cong Q^{(y)}(5)_1$:

$$U_2 = \langle x^2, x^3 \rangle \in Q(0); \; W_2 = \langle yx^2, yx^3 \rangle \in Q(0).$$

Also, $Q^{(y)}(6) \cong U_1 = W_1$ satisfies

$$U_1 = \langle z^2, x^2z \rangle \in Q(1).$$

Also $(A, x + y + z)$ is SL, as $P_{x+y+z} = (7, 5, 4, 2, 2, 2) = H^\vee.$
The convention we use in Table 3.1 is to exhibit the new elements compared to the adjacent two smaller squares: thus in the box for \(((y^2) + (0 : y^3))\) we’ll write representatives in \(Q(0)\) or \(Q(1)\) for elements of \((y^2) + (0 : y^3)\) not seen in the adjacent smaller submodules of \(A^*\), \(((y^3) + (0 : y^3))\) or \(((y^2) + (0 : y))\).

<table>
<thead>
<tr>
<th>$+$</th>
<th>1</th>
<th>((y))</th>
<th>((y^2))</th>
<th>((y^3))</th>
<th>0</th>
<th>(Q(0))</th>
<th>(Q(1))</th>
</tr>
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<tr>
<td>0</td>
<td>(1, x, x^2, x^3, x^4, x^5)</td>
<td>(y, xy, x^2y, x^3y)</td>
<td>(x^4, x^5)</td>
<td>(zy^3, xzy^3)</td>
<td>0</td>
<td>(Q(0))</td>
<td>(Q(1))</td>
</tr>
<tr>
<td>((0 : y))</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>(x^2y, x^3y, x^4y, x^5y)</td>
<td>(Q(0))</td>
<td>(z^2, y^3, x^2z, xy^3)</td>
<td>(Q(1))</td>
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<tr>
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<td>–</td>
<td>–</td>
<td>(x^2, x^3, x^4, x^5)</td>
<td>(Q(0))</td>
<td>(y^2, xy^2)</td>
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<td>((0 : y^3))</td>
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<td>(y, xy)</td>
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<td>(A^*)</td>
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<td>1, (x)</td>
<td>(Q(0))</td>
<td>(Q(1))</td>
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</table>

Table 3.1: \((0 : y^a) + (y^b)\) for \(I = \text{Ann} (F), F = x^5y + x^2z^3 + xy^3z\)

**Apply?** What is the Jordan block matrix of a generic element of \(\mathcal{N}(z) \subseteq \mathcal{C}(z)\), nilpotents in the centralizer of \(z\)? One finds \(B\) generic, \(B \in \pi^{-1}(\prod \text{(nilpotents)} \subseteq \prod \text{End}(U_i)).\) What are Jordan blocks for \(B^*\) (open, partial results: P. Oblak, T. Košir, G. McNinch, D. I. Panyushev, R. Basili and I.- [Bas2, Ob1, Ob2, KO, Pan, McN]).
Q. What is the goal of this work? Do you expect to characterize Gorenstein Artin algebras?

Ans. We wish to characterize more properties of the associated graded algebras, and also the Hilbert functions for Gorenstein (non-graded) algebras. The invariants we discuss can also be deformation invariants within the family $Gor(H)$ parametrizing AG algebras of Hilbert function $H$ [I].

Artinian Gorenstein algebras can occur in singularity theory: for a finite mapping germ, the isomorphism class of the associated Artinian algebra is a right-left invariant.

Homogeneous AG algebras are understood for three variables, due to the Buchsbaum-Eisenbud Pfaffian structure theorem, and in height threee, for symmetric Gorenstein sequences $H$, the family $Gor(H)$ is irreducible; but in height four, $Gor(H)$ may have many irreducible components. The possible $H$ are known in heights no greater than three, but have not been characterized in height four (See [MNZ]).

For non-graded GA algebras, even in height three the possible $H$ are not known, despite the presence of a structure theorem (see [I]).

References


