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Research Statement

My research lies at the intersection of algebraic geometry and combinatorics. In particular, I study Schubert varieties, their conormal varieties, Steinberg varieties, Springer fibres, and orbital varieties. These objects are closely related via Springer theory. Additionally, I am also interested in the study of hyperplane and subspace arrangements associated to finite linear groups.

In joint work with Lakshmibai and Seshadri, I have studied the cotangent bundles of flag varieties [LSS17], and the conormal varieties of their Schubert subvarieties [LS17b, LS17a, Sin18]. The key objects are the irreducible components of Springer fibres, orbital schemes, and Steinberg varieties; these are all closely related to the conormal variety of a Schubert variety. In many cases, these irreducible components are indexed by classical combinatorial objects such as Young tableaux and Weyl groups. A central motif in the subject is that these components are usually described as closures of certain affine cells, but an explicit description of the boundary is missing in most cases. An overarching theme of my research is to describe this boundary explicitly by presenting systems of defining equations. Simultaneously, I intend to study the local geometry of these objects; more ambitiously, I would also like to develop standard monomial theories, and deduce toric degenerations and Gröbner bases for these objects.

I have also studied the combinatorics of subspace arrangements associated to finite linear groups. This is a joint project with Martino [MS18]. We show that for any finite $G \subset SL_n(\mathbb{R})$ generated by rotations, the intersection lattice $\mathcal{A}^G \stackrel{\text{def}}{=} \{U \subset \mathbb{R}^n \mid Stab_G(U) \neq \{e\}\}$ is atomic. Further, we compute $\mathcal{A}^G$ for all finite subgroups of $GL_3(\mathbb{R})$. We intend to carry out a similar analysis for complex reflection and rotation groups.

Recollections

I first present a quick recollection on certain aspects of the relationships between Schubert varieties, their conormal varieties, Steinberg varieties, orbital varieties, and the Springer map. Throughout, G will be a connected simple algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. We fix a Borel subgroup $B \subset G$ and a compact manifold X with a transitive G action.

1. Let $\mathcal{N}$ denote the nilpotent cone, i.e., the variety of nilpotent elements in $\mathfrak{g}$. Every G-orbit $\mathcal{N}_\lambda \subset \mathcal{N}$ admits a canonical symplectic form, called the *Kirillov-Kostant* form.
2. Consider the fibre bundle $\mathcal{X} = G \times^B X \to {}^G/_B$ given by $(g, x) \mapsto gB$. The G-orbit closures in $\mathcal{X}$ are precisely the subvarieties $G \times^B X_w$, where X_w is a Schubert subvariety of X.
3. The *Steinberg variety* is the union of the conormal varieties of the G-orbit closures in $\mathcal{X}$. Each irreducible component of the Steinberg variety is a fibre bundle $G \times^B T^*_X X_w \to {}^G/_B$, where $T^*_X X_w$ denotes the *conormal variety* of a Schubert variety $X_w \subset X$, see [CG97, Ste88].
4. The G-action on X admits a moment map $T^*X \to \mathfrak{g}^*$. Identifying $\mathfrak{g} \cong \mathfrak{g}^*$ via the *Killing form*, we obtain a map $T^*X \to \mathfrak{g}$, whose image is some G-orbit closure $\overline{\mathcal{N}_\lambda} \subset \mathcal{N} \subset \mathfrak{g}$. The induced map $\mu : T^*X \to \overline{\mathcal{N}_\lambda}$, called the *Springer map*, is a proper and finite symplectomorphism.
5. If μ is birational and $\overline{\mathcal{N}_\lambda}$ is normal, then μ is a symplectic resolution, see [Fu03, Pan91]. When μ is a symplectic resolution, the *Springer fibre* $\mu^{-1}(x)$ is connected and pure dimensional, see [CG97, Spa82]. While it is known that the irreducible components of a Springer fibre are generally singular, see [FM10], the geometry of Springer fibres seems to be little understood overall, see [HR$^+$].
6. Let $\mathfrak{u}$ denote the Lie algebra of the unipotent radical of B. Consider the variety $\mathcal{O}_\lambda = \mathcal{N}_\lambda \cap \mathfrak{u}$, where $\mathcal{N}_\lambda$ is some G-orbit in $\mathcal{N}$. The closure $\overline{\mathcal{O}_\lambda}$ of $\mathcal{O}_\lambda$ in $\mathfrak{u}$ is called an orbital scheme, and the irreducible components of $\overline{\mathcal{O}_\lambda}$ are called *orbital varieties of shape* λ.

7. For any Schubert variety $X_w \subset X$, the image of its conormal variety under the Springer map is an orbital variety, i.e., $\mu(T^*_X X_w)$ is an orbital variety. The fibre of the induced map $\mu|T^*_X X_w$ over a generic point x is an irreducible component of the Springer fibre $\mu^{-1}(x)$.

Schubert Divisors and the Minimal Nilpotent Orbit

Schubert subvarieties $X_w \subset X$ are indexed by certain Weyl group elements; there is a natural choice of indexing for which we have $\operatorname{codim} X_w = l(w)$, where $l(w)$ denotes the length of w. In particular, X_w is a divisor if and only if $w = s_\alpha$ for some simple root α. If X_{s_α} is a divisor corresponding to a *long root* α, then $\mu(T^*_X X_{s_\alpha})$ is an orbital variety of shape *min*, where $\mathcal{N}_{min}$ is the minimal non-zero G-orbit in $\mathcal{N}$.

Generalizing a result of Lusztig [Lus81], Achar and Henderson [AH13] have constructed an open embedding $\psi : \mathcal{N}_{min} \hookrightarrow X_\epsilon$ of the minimal orbit into a Schubert subvariety of the affine Grassmannian of G. Together with Lakshmibai, I have extended this result.

Theorem 1 ([LSS17, Sin19])**.** *Let X_{s_α} be a Schubert divisor corresponding to a* long root *α.*

1. *Let O be an orbital variety of shape min. Then $\psi(O)$ is a Schubert subvariety of X_ϵ.*
2. *Let $\mathcal{G}$ denote the loop group $G[t, t^{-1}]$. There exists an open embedding $\widetilde{\psi} : T^*_X X_{s_\alpha} \hookrightarrow X_\eta$ lifting ψ, with X_η a Schubert subvariety of $\mathcal{G}/\mathcal{P}$, where $\mathcal{P}$ is a two-sep parabolic subgroup.*
3. *Let π denote the projection from $\mathcal{G}/\mathcal{P}$ to the affine Grassmannian. For any X_{s_α} as above, we have a commutative diagram,*

$$\begin{array}{ccc} T^*_X X_{s_\alpha} & \overset{\widetilde{\psi}}{\hookrightarrow} & X_\eta \\ \downarrow{\scriptstyle \mu} & \times & \downarrow{\scriptstyle \pi} \\ \mu(T^*_X X_{s_\alpha}) & \overset{\psi}{\hookrightarrow} & X_\zeta \, . \end{array}$$

4. *The fibre of the map $T^*_X X_w \to \mu(T^*_X X_w)$ is isomorphic to an open subset of a Schubert variety.*

Corollary 2. *Let α be a long root, and x an element of the $\mathcal{N}_{min}$. Suppose Y is either conormal variety of a Schubert divisor X_{s_α}, an orbital variety of shape min, or an irreducible component of the Springer fibre $\mu^{-1}(x)$. Then Y is normal, Cohen-Macaulay, Frobenius split, and admits a resolution of singularities via an open subvariety of a Bott-Samelson variety.*

Perrin and Smirnov [PS12] have obtained a similar result on the components of a Springer fibre: Suppose G is either SL_n or SO_{2n}. Then for any $x \in \mathcal{N}$ satisfying $x^2 = 0$, the irreducible components of $\mu^{-1}(x)$ can be identified as open subsets of some Schubert variety.

Cominuscule Schubert Varieties

Let θ denote the highest root in the root system of G. A simple root α is called *cominuscule* if the coefficient of α in θ is one. Let P be a parabolic subgroup of G corresponding to 'omitting' a cominuscule root. The space $X = G/P$ is called a *cominuscule Grassmannian*, and its Schubert subvarieties $X_w \subset X$ are called cominuscule Schubert varieties.

In [LRS16], Lakshmibai *et al.* constructed a compactification $\phi : T^*X \hookrightarrow X_\kappa$ of the cotangent bundle T^*X of a cominuscule Grassmannian X into an *affine Schubert variety* X_κ. Together with Lakshmibai, I have extended this result to algebraically closed fields of *good characteristic*, and studied the image of the conormal varieties $T^*_X X_w$ under the map ϕ.

Theorem 3 ([LRS16, LS17a])**.** *The closure $\overline{\phi(T^*_X X_w)}$ is a Schubert subvariety of X_κ if and only if the* opposite *Schubert variety X^-_w is smooth.*

Under the compactification ϕ, the Springer map μ and the structure map $\pi : T^*_X X_w \to X$ are both naturally identified as (the restriction to X_κ of) projections from an affine flag variety to an affine Grassmannian. Inspired by this, I have conjectured the following.

Conjecture 4. *For any cominuscule Schubert variety $X_w \subset X$, we have,*

$$T^*_X X_w = \pi^{-1}(X_w) \bigcap \mu^{-1}(\mu(T^*_X X_w)).$$

Recall that for $G = SL_n$, the G-orbits $\mathcal{N}_\lambda \subset \mathcal{N}$ are indexed by partitions $\lambda \vdash n$. Given a *standard Young tableau* τ, let τ_i denote the sub-diagram of τ supported on the boxes labeled $1, \cdots, i$. For $x \in \mathcal{N}$, let $J_i(x)$ denote the *Jordan type* of the top left $i \times i$ submatrix of x. Consider the variety

$$\mathcal{O}_\tau = \{x \in \mathcal{N} \mid J_i(x) = \tau_i\}.$$

The orbital varieties of SL_n are precisely the closures $\overline{\mathcal{O}_\tau}$. In particular, the orbital varieties of shape λ are indexed by standard Young tableaux of shape λ.

Given a standard Young tableau τ with exactly two columns, there exists some Schubert subvariety X_w of some Grassmannian $X = Gr(d, n)$, such that $\overline{\mathcal{O}_\tau}$ is the birational image of $T^*_X X_w$ under μ. I have verified Conjecture 4 set-theoretically in types A, B, C. In the process, I have also obtained, defining equations for the corresponding orbital varieties, up to reducedness. The defining equations for these orbital varieties were independently obtained earlier in [Mel05, BM17].

Theorem 5 ([Sin18])**.** *Conjecture 4 holds set-theoretically in types A, B, C. Further, there exist integers r_{ij}, depending on w, such that*

$$\mu(T^*_X X_w) = \left\{ x \in \mathcal{N} \,\middle|\, x^2 = 0,\, rk(x_i^j) \leq r_{ij} \ \forall 1 \leq i < j \leq n \right\},$$

where x_i^j denotes the south-west sub-matrix of x whose north-east corner is (i, j).

The key step in my proof of Theorem 5 is the construction of a resolution of singularities $\widetilde{Z_w} \to T^*_X X_w$ for the conormal variety of a cominuscule Schubert variety.

Let X be any compact G-homogeneous space, and $\widetilde{X_w} \to X_w$ a Bott-Samelson resolution of a Schubert subvariety $X_w \subset X$. Recall the conormal bundle of the Schubert cell, $T^*_X X^\circ_w \to X^\circ_w$. We have composite maps $T^*_X X^\circ_w \to X^\circ_w \hookrightarrow X$ and $\widetilde{X_w} \to X_w \hookrightarrow X$, and fibre products,

$$Z^\circ_w := T^*_X X^\circ_w \times_X \widetilde{X_w}, \qquad Y_w := T^*X \times_X \widetilde{X_w}.$$

We identify Z°_w as a locally closed subvariety of Y_w, and set $Z_w = \overline{Z^\circ_w}$, the closure of Z°_w in Y_w. One easily sees that Z_w admits a proper, birational map $Z_w \to T^*_X X_w$.

Theorem 6 ([Sin18])**.** *Suppose X_w is a cominuscule Schubert variety. Then Z_w is smooth; in particular, the map $Z_w \to T^*_X X_w$ is a resolution of singularities.*

Further, if the map $T^*_X X_w \xrightarrow{\mu} \mu(T^*_X X_w)$ is birational, the composite map $Z_w \to \mu(T^*_X X_w)$ is also a resolution of singularities. Consequently, we obtain in type A, a resolution of singularities for all orbital varieties $\overline{\mathcal{O}_\tau}$ corresponding to standard Young tableaux τ with precisely two columns.

Future Work

I describe here some open problems in the subject along with some possible approaches.

Problem 7. *Construct a small resolution of $T^*_X X_w$, the conormal variety of a Schubert variety.*

Recall that a resolution of singularities $\pi : Y \to X$ is called *small* if we have the inequalities,

$$\operatorname{codim}_X \left\{ x \in X \,\middle|\, \dim(\pi^{-1}(x)) = k \right\} > 2k, \quad \forall k.$$

In [Zel83], Zelevinsky constructed a small resolution for the Schubert subvarieties of $Gr(d, n)$. This work has been generalized to orthogonal and Lagrangian Grassmannians by Sankaran and Vanchinathan [SV94], and to all minuscule Grassmannians by Perrin [Per07]. Suppose we replace the Bott-Samelson variety with a small resolution in the construction of Z_w. Does this yield a small resolution of $T^*_X X_w$? □

Problem 8. *Let X be a minuscule or cominuscule Grassmannian, and let $\mathcal{L}_w$ denote the irreducible $\mathcal{D}$-module with regular singularities on X supported on a Schubert subvariety $X_w \subset X$. Is the singular support of $\mathcal{L}_w$ irreducible?*

Bressler, Finkelberg, and Lunts [BFL90] have shown that this is true in type A. The key ingredient in their proof is the micro-local fibre of Zelevinsky's [Zel83] small resolution. I believe that the generalizations of [Zel83] given in [SV94, Per07], along with my description [Sin18] of the conormal variety, provides the ingredients necessary to develop a type-independent generalization of the proof in [BFL90]. □

Problem 9. *Present a system of defining equations for conormal varieties, orbital varieties, and Springer components, and describe their boundaries.*

Recall that any compact homogeneous space is of the form $X = {}^{G}/_{P}$ for some parabolic subgroup $P \subset G$. We can identify its cotangent bundle as $T^*X = G \times^P \mathfrak{u}_P$, where $\mathfrak{u}_P$ is the Lie algebra of the unipotent radical of P. For $\widetilde{X_w} = {}^{P_1 \times^B \cdots \times^B P_k}/_{B}$ a Bott-Samelson variety resolving the Schubert variety $X_w \subset X$, the variety Y_w, see Pg 4, has the following description,

$$Y_w = P_1 \times^B \cdots \times^B P_k \times^B \mathfrak{u}_P \to \widetilde{X_w} \to X_w.$$

Let ${}^{\mathcal{N}}/_{B}$ denote the set of orbits in $\mathcal{N}$. For $1 \leq i \leq k$, we can define maps $\pi_i : Y_w \to {}^{\mathcal{N}}/_{B}$, given by $(p_1, \cdots, p_k, x) \mapsto Ad(p_i \cdots p_k)x$. □

Conjecture 10. *There exist B-stable linear subspaces $\mathfrak{u}_i \subset \mathfrak{u}$, for $1 \leq i \leq k$, such that,*

$$Z_w = \bigcap_{1 \leq i \leq k} \pi_i^{-1}({}^{\mathfrak{u}_i}/_{B}).$$

Let us denote the right hand side by Z'_w. The correct choices for $\mathfrak{u}_i$ are easy to determine, and for these $\mathfrak{u}_i$, it is clear that Z_w is an irreducible component of Z'_w. Conversely, for $G = SL_n$, calculations indicate that Z'_w can be constructed via a sequence of blowups and $\mathbb{P}^1$ fibrations, hence is irreducible. □

Conjecture 10 allows us to view $T^*_X X_w$ (and the orbital variety $\mu(T^*_X X_w)$) as the image (under a proper map) of a combinatorially described subvariety, giving us a handle on its boundary and defining equations. It also opens up an approach to the following problem.

Problem 11. *Consider the set of all orbital varieties for fixed G, B, but arbitrary shape. Give a combinatorial description of the inclusion order on the set of orbital varieties.*

Joseph and Hinich [HJ05] have described this inclusion order "in terms of certain (as yet unknown) geometric data", by relating it to the Borel-Moore homology of the Steinberg variety.

Assuming Conjecture 10 to be true, one can generate a system of defining equations for the orbital varieties of SL_n. I have verified using Sage that the defining equations for the orbital varieties of SL_n implied by Conjecture 10 are indeed correct for $n \leq 7$. □

Remark. *A variant of Conjecture 10 asks for a combinatorial description of the generic fibre of the composite map $Z_w \to T^*_X X_w \xrightarrow{\mu} \mathcal{N}$. A satisfying answer to this will yield information about the boundary of Springer components.*

Problem 12. *Do the equations in Theorem 5 generate a radical ideal? More generally, find a Gröbner basis for the corresponding radical ideal.*

Theorem 5 allows us to view a two column orbital variety as the intersection of the *matrix Schubert variety* given by the rank conditions, and the subvariety given by $x^2 = 0$, which is generated by quadratic equations. The monomial theory of matrix Schubert varieties has been studied in detail, see [KM05], and is possibly a good starting point for this question. □

Problem 13. *Can we extend Theorem 1 to non-minimal orbits along the lines of [AH13]?*

In [AH13], Achar and Henderson exhibit a subvariety of the affine Grassmannian of G that is closely related to $\mathcal{N}$. I plan to explore if this extends to the orbital subvarieties in $\mathcal{N}$. □

In conclusion, the primary goal of my research is to understand the geometry of conormal varieties, orbital varieties, and Springer components. Going forward, I plan to explore their boundaries, singular loci, normality, rational smoothness etc., and search for Gröbner bases and semi-toric degenerations.

References

[AH13] Pramod N. Achar and Anthony Henderson, *Geometric Satake, Springer correspondence and small representations*, Selecta Math. (N.S.) **19** (2013), no. 4, 949–986. MR 3131493

[BFL90] P. Bressler, M. Finkelberg, and V. Lunts, *Vanishing cycles on Grassmannians*, Duke Math. J. **61** (1990), no. 3, 763–777. MR 1084458

[BM17] Nurit Barnea and Anna Melnikov, *B-orbits of square zero in nilradical of the symplectic algebra*, Transform. Groups **22** (2017), no. 4, 885–910. MR 3717217

[CG97] Neil Chriss and Victor Ginzburg, *Representation Theory and Complex Geometry*, Birkhäuser Boston, Inc., Boston, MA, 1997. MR 1433132

[FM10] Lucas Fresse and Anna Melnikov, *On the singularity of the irreducible components of a Springer fiber in $\mathfrak{sl}_n$*, Selecta Math. (N.S.) **16** (2010), no. 3, 393–418. MR 2734337

[Fu03] Baohua Fu, *Symplectic resolutions for nilpotent orbits*, Invent. Math. **151** (2003), no. 1, 167–186. MR 1943745

[HJ05] Vladimir Hinich and Anthony Joseph, *Orbital variety closures and the convolution product in Borel-Moore homology*, Selecta Math. (N.S.) **11** (2005), no. 1, 9–36. MR 2179652

[HR+] Jim Humphreys, Ryan Reich, et al., *Smoothness properties of the Springer fiber*, MathOverflow, mathoverflow.net/q/112415.

[KM05] Allen Knutson and Ezra Miller, *Gröbner geometry of Schubert polynomials*, Ann. of Math. (2) **161** (2005), no. 3, 1245–1318. MR 2180402

[LRS16] V. Lakshmibai, Vijay Ravikumar, and William Slofstra, *The Cotangent Bundle of a Cominuscule Grassmannian*, Michigan Math. J. **65** (2016), no. 4, 749–759. MR 3579184

[LS17a] V. Lakshmibai and Rahul Singh, *Conormal Varieties on the Cominuscule Grassmannian*, arXiv preprint 1712.06737 (2017).

[LS17b] ———, *Cotangent Bundles of Partial Flag Varieties and Conormal Varieties of their Schubert Divisors*, arXiv preprint 1712.00107 (2017).

[LSS17] V. Lakshmibai, C. S. Seshadri, and Rahul Singh, *Cotangent Bundle to the Flag Variety – I*, Transformation Groups (2017).

[Lus81] G. Lusztig, *Green Polynomials and Singularities of Unipotent Classes*, Adv. in Math. **42** (1981), no. 2, 169–178. MR 641425

[Mel05] Anna Melnikov, *The combinatorics of orbital varieties closures of nilpotent order 2 in* sl_n, Electron. J. Combin. **12** (2005), Research Paper 21, 20. MR 2134184

[MS18] Ivan Martino and Rahul Singh, *Finite Groups Generated in Low Real Codimension*, arXiv preprint, 1804.05089 (2018).

[Pan91] D. I. Panyushev, *Rationality of singularities and the Gorenstein property of nilpotent orbits*, Funktsional. Anal. i Prilozhen. **25** (1991), no. 3, 76–78. MR 1139878

[Per07] Nicolas Perrin, *Small resolutions of minuscule Schubert varieties*, Compos. Math. **143** (2007), no. 5, 1255–1312. MR 2360316

[PS12] Nicolas Perrin and Evgeny Smirnov, *Springer Fiber Components in the Two Columns case for types A and D are normal*, Bull. Soc. Math. France **140** (2012), no. 3, 309–333. MR 3059118

[Sin18] Rahul Singh, *Conormal Varieties on the Cominuscule Grassmannian – II*, arXiv preprint 1805.12297 (2018).

[Sin19] ———, *Doctoral thesis*, Ph.D. thesis, Northeastern University, 2019.

[Spa82] Nicolas Spaltenstein, *Classes unipotentes et sous-groupes de Borel*, Lecture Notes in Mathematics, vol. 946, Springer-Verlag, Berlin-New York, 1982. MR 672610

[Ste88] Robert Steinberg, *An occurrence of the Robinson-Schensted correspondence*, J. Algebra **113** (1988), no. 2, 523–528. MR 929778

[SV94] Parameswaran Sankaran and P. Vanchinathan, *Small resolutions of Schubert varieties in symplectic and orthogonal Grassmannians*, Publ. Res. Inst. Math. Sci. **30** (1994), no. 3, 443–458. MR 1299524

[Zel83] A. V. Zelevinskiĭ, *Small resolutions of singularities of Schubert varieties*, Funktsional. Anal. i Prilozhen. **17** (1983), no. 2, 75–77. MR 705051