Plan of research: Inverse problems for Maxwell’s equations.

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Under the guidance of my advisor, Prof. Ting Zhou, I have started working on several inverse problems in electrodynamics.

Inverse problems arise naturally in many physical contexts, where one wishes to obtain information about an object by means of indirect measurements. Applications include medical imaging, nondestructive testing of materials, and seismology, and the goal is to reconstruct material properties by nonintrusive measurements taken on the surface of the body, or at a distance from the body. This wide range of applications makes inverse problems a very actively researched area in Mathematics with strong ties to many engineering disciplines.

A seminal work in the mathematical study of inverse problems is a paper by A. P. Calderón [Cal80], who had previously worked in the oil industry and posed the question whether it is possible to obtain the electrical conductivity of a medium from voltage and current measurements taken at the boundary. This problem, which became known as Calderón’s problem, is the inverse problem for the conductivity equation in a domain $\Omega \subset \mathbb{R}^n$,

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega,$$

where the parameter $\sigma$ is the electrical conductivity. The direct problem consists in finding the solution $u$, given $\sigma$ as well as some boundary data, $u|_{\partial \Omega} = f$. The goal of the inverse problem on the other hand is to determine $\sigma$, given the voltage-to-current or Dirichlet-to-Neumann map on the boundary,

$$\Lambda_\sigma(f) = \left( \frac{\partial u}{\partial \nu} \right)|_{\partial \Omega},$$

where $\nu$ is the outer unit normal on $\partial \Omega$, and $u$ is the solution to (1) with $u|_{\partial \Omega} = f$. That is, knowing the current on the boundary $(\sigma \frac{\partial u}{\partial \nu})|_{\partial \Omega}$ induced by any given voltage potential $f$ on the boundary, find the conductivity $\sigma$ of the material. In his paper, Calderón considered a linearized version of this problem. The problem was then taken up by several authors. In [KV84], Kohn and Vogelius showed that the Dirichlet to Neumann map determines the conductivity on the boundary. The breakthrough came with the work [SU87], in which Sylvester and Uhlmann showed that a smooth conductivity can be recovered from knowledge of the map (2). The proof was based on the construction of the so-called complex geometrical optics solutions (CGO solutions), that are functions of the form

$$v(x) = e^{i\varepsilon x} \left( 1 + r(\xi, x) \right),$$

where $|\xi|$ is large and $r$ decreases in some sense as $|\xi| \to \infty$. Their method has since been adapted to show results for less regular conductivities, see e.g. [HT13, CR16], as well as for other equations, such as the time-harmonic Maxwell’s equations in electrodynamics [OS96, CZ14].

The framework for the electrodynamic inverse problem is as follows: We consider the time-harmonic Maxwell’s equations with frequency $\omega$ for the electric field $E$ and magnetic field $H$ in a domain $\Omega \subset \mathbb{R}^3$,

$$\nabla \times E - i \omega \mu H = 0, \quad \nabla \times H + i \omega \gamma E = 0 \quad \text{in } \Omega,$$

with a tangential boundary condition of the form $\nu \times E|_{\partial \Omega} = f$. Here, $\nabla \times$ denotes the curl in $\mathbb{R}^3$; $\gamma = \varepsilon + i\sigma/\omega$; and the electric permittivity $\varepsilon$, the magnetic permeability $\mu$ and the conductivity $\sigma$ are nonnegative functions.

The direct problem is to find $E$ and $H$, given $\mu, \varepsilon, \sigma$, and boundary data. The inverse problem consists in reconstructing the material parameters $\mu, \varepsilon, \sigma$, given some knowledge of the behavior of $E$ and $H$ on the boundary. I am studying two particular situations which I will describe in more detail below.
1 Inverse problem on a bounded Lipschitz domain with full boundary data.

In this scenario, $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary. It is known that for a given set of Lipschitz continuous parameters $\mu$, $\varepsilon$, and $\sigma$, and tangential boundary data in a suitable space (here, $\text{Div}F$ denotes the surface divergence),

$$\nu \wedge E = f \in TH^{1/2}_{\text{Div}}(\partial\Omega) = \left\{ F \in H^{1/2}(\partial\Omega)^3 \mid \nu \cdot F = 0, \, \text{Div}F \in H^{1/2}(\partial\Omega) \right\} \text{ on } \partial\Omega,$$

the equations (4) have a unique solution pair $(E, H)$ in the space $H_{\text{curl}}(\Omega) \times H_{\text{curl}}(\Omega)$, where

$$H_{\text{curl}}(\Omega) = \left\{ u \in L^2(\Omega)^3 : \nabla \wedge u \in L^2(\Omega) \right\}.$$

For the inverse problem, for a given frequency $\omega$, suppose we know the so-called Cauchy data set

$$C(\mu, \varepsilon, \sigma; \omega) = \left\{ (\nu \wedge E|_{\partial\Omega}, \nu \wedge H|_{\partial\Omega}) : (E, H) \text{ solves (4) with parameters } \mu, \varepsilon, \sigma \right\}. \tag{7}$$

That is, we know the tangential components of any pair of functions $(E, H) \in H_{\text{curl}}(\Omega)^2$ that solve (4) almost everywhere in $\Omega$. The question is whether this set of data uniquely determines $\mu$, $\varepsilon$, and $\sigma$.

This problem has previously been treated in several instances given higher regularity of the domain and parameters; notably, in [OS96], uniqueness of the solution to the inverse problem has been shown for smooth parameters on a smooth domain; more recently, in [CZ14] the case of continuously differentiable parameters on a domain with $C^1$ boundary was examined. On the other hand, for the conductivity equation, uniqueness has been shown even for Lipschitz continuous parameters [CR16]; both [CR16] and [CZ14] made use of an averaged estimate that was first derived in [HT13], where the authors studied the inverse problem for the conductivity equation on a Lipschitz domain with $C^1$ conductivity.

My aim is to adapt the methods used in [CR16] to the situation of Maxwell’s equations and combine them with techniques for Maxwell’s equations [OS96, CZ14] in order to be able to show that the present inverse problem is uniquely solvable in the case of Lipschitz parameters.

To achieve this, the first step is to transform the problem into a vector Schrödinger equation of the form

$$[-(\Delta + k^2) + Q(\mu, \varepsilon, \sigma)]X = 0 \tag{8}$$

by adding two redundant equations and rescaling the functions suitably. This leaves us with an elliptic system, for which we have the rich theory of elliptic differential operators readily available. This transformation was introduced in [OS96]; due to the lower regularity in the present case, we obtain a system with a weakly defined potential, such that we will seek for weak solutions. After constructing solutions to this system, it is important to assure that the solutions found this way will yield feasible solutions to the original Maxwell system.

As common in uniqueness proofs for inverse problems, we will next assume two sets of parameters $(\mu_1, \varepsilon_1, \sigma_1)$ and $(\mu_2, \varepsilon_2, \sigma_2)$ have identical Cauchy sets, and derive a certain integral identity of the form

$$\langle (Q(\mu_2, \varepsilon_2, \sigma_2) - Q(\mu_1, \varepsilon_1, \sigma_1))w_1, v_2 \rangle = 0, \tag{9}$$

where $w_1$ solves the vector Schrödinger equation (8) with potential $Q(\mu_1, \varepsilon_1, \sigma_1)$, and $v_2$ solves a related first order elliptic equation involving parameters $(\mu_2, \varepsilon_2, \sigma_2)$.

The next step is to construct a sufficient number of CGO solutions $w_1$ and $v_2$ such that eventually from (9) we will be able to conclude that in fact $(\mu_2, \varepsilon_2, \sigma_2) = (\mu_1, \varepsilon_1, \sigma_1)$. In this step we will use a generalization of an a priori estimate derived in [CR16], which will facilitate the use of the Riesz representation theorem to obtain the desired functions. The construction is complicated by the fact that the functions need to be chosen carefully so as to not only solve the respective elliptic equations, but to also yield solutions to Maxwell’s equations. For this to work, one needs a uniqueness result for the solution constructed for the reduced Schrödinger equation.

Finally, by plugging the CGO solutions into (9) and taking the limit of the large parameter $|\xi|$, we will obtain certain relations between the parameters that allow us to conclude that the two sets of parameters are equal and thus complete the proof.
2 Inverse problem on an unbounded domain with partial data.

The second problem that I have been studying is a partial data problem, i.e., one in which boundary data is available only on a subset of the boundary. Such problems arise naturally in applications where it is not possible to take measurements on the whole surface of a given body, for example when part of the boundary is not accessible, or when it would simply be too costly to do so.

The setting for the problem that I am working on is that of a slab, an infinite domain bounded by two parallel planes:

$$\Omega = \{ x \in \mathbb{R}^3 : 0 < x_3 < L \}.$$ 

This situation models a body with significantly smaller dimensions in one direction than the other two; a typical application of this geometry is the modeling of wave propagation in shallow water.

The other main difference to the problem discussed in Section 1 aside from having only partial data available is the fact that the domain $\Omega$ now is unbounded. This makes it necessary to impose additional requirements upon the solutions concerning their behavior at infinity in order to guarantee unique solvability of the direct problem: a suitable radiation condition is needed.

There are a number of results on partial data problems in a slab for the conductivity and Schrödinger equations: In [LU10], the authors study these equations in the two cases that partial Dirichlet and Neumann data are given (i) on the same boundary hyperplane or (ii) on opposite boundary hyperplanes. The complication when compared to full data boundary value problems is that in the process of deriving an integral formula, one obtains boundary integrals with unknown boundary terms. Thus, additional work is required to control these terms. The authors use a Carleman estimate to show that these boundary integrals are negligible in case (i); in case (ii), special CGO solutions that vanish on one boundary hyperplane are constructed using a reflection argument. In [KLU12], a Schrödinger operator with a magnetic term was studied assuming the same geometry, using reflection arguments to construct the necessary CGOs.

The setting for the inverse problem for Maxwell’s equations that I want to study is the following: Let $\Omega$ be the slab described above, and denote $\Gamma_1 = \{ x_3 = L \}$ and $\Gamma_2 = \{ x_3 = 0 \}$ the boundary hyperplanes; furthermore, let $\Gamma'_1 \subset \Gamma_1$ and $\Gamma'_2 \subset \Gamma_2$ be open subsets. We consider the time-harmonic Maxwell equations

$$\nabla \wedge E - i \omega \mu H = 0, \quad \nabla \wedge H + i \omega \gamma E = 0 \text{ in } \Omega, \quad \text{suitable radiation condition for } E \text{ and } H \text{ as } |(x_1, x_2)| \to \infty,$$

(10)

where $\mu$ and $\gamma = \varepsilon + i \sigma/\omega$ are functions that are non-constant only in a compact set $K$, so that $\mu \equiv \mu_o > 0$, $\varepsilon \equiv \varepsilon_o > 0$ and $\sigma \equiv 0$ outside $K$. Furthermore, we are given some partial Cauchy data. We want to investigate cases analogous to those considered in [LU10, KLU12]: on the one hand, knowledge of the tangential boundary components $\nu \wedge E$ and $\nu \wedge H$ on opposite boundary hyperplanes, and on the other hand, knowledge of the boundary values on the same boundary hyperplane. Consequently, the partial Cauchy data set will be of the form

$$C(\mu, \varepsilon, \sigma) = \{(\nu \wedge E)|_{\Gamma_1}, (\nu \wedge H)|_{\Gamma'_2} : (E, H) \text{ solves (10)}\} \quad \text{(11)}$$

in the first case, and in the latter case

$$C(\mu, \varepsilon, \sigma) = \{(\nu \wedge E)|_{\Gamma_1}, (\nu \wedge H)|_{\Gamma'_1} : (E, H) \text{ solves (10)}\}. \quad \text{(12)}$$

The question is whether knowledge of the Cauchy data set in either case is sufficient to guarantee uniqueness of the parameters in $\Omega$.

In the treatment of this problem, the first step will be to show the well-posedness of the direct problem, which hasn’t been studied before. This will be done using the Lax-Phillips method to split the problem into one with constant coefficients in the whole slab and one with non-constant coefficients in a bounded subdomain of the slab. In the course of this, we will also formulate the radiation condition that is needed to obtain a unique solution.
Then, for each of the situations described above in the formulation of the inverse problem, we will derive a suitable integral identity involving the unknown parameters, which will involve using Carleman estimates to control certain boundary terms, and then proceed to construct CGO solutions. To do so, reflection arguments will be needed. Since by reflecting the parameters along the boundary, one obtains parameters that are only Lipschitz continuous in general, at this point the problem connects with the situation described in Section 1 and results obtained in that setting will be employed. The final step will be to plug the CGO solutions into the integral identity and take a limit to obtain relations between the parameters that will allow to prove their uniqueness.

References


