Two inverse problems for Maxwell’s equations

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1 Introduction

Inverse problems arise naturally in many physical contexts, where one wishes to obtain information about an object by means of indirect measurements. Applications include medical imaging, nondestructive testing of materials, and seismology, and the goal is to reconstruct material properties by nonintrusive measurements taken on the surface of the body, or at a distance from the body. This wide range of applications makes inverse problems a very actively researched area in Mathematics with strong ties to many engineering disciplines.

A seminal work in the mathematical study of inverse problems is a paper by A. P. Calderón [Cal80], who had previously worked in the oil industry and posed the question whether it is possible to obtain the electrical conductivity of a medium from voltage and current measurements taken at the boundary. This problem, which became known as Calderón’s problem, is the inverse problem for the conductivity equation in a domain $\Omega \subset \mathbb{R}^n$,

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega,$$

where the parameter $\sigma$ is the electrical conductivity. The direct problem consists in finding the solution $u$, given $\sigma$ as well as some boundary data, $u|_{\partial \Omega} = f$. The goal of the inverse problem on the other hand is to determine $\sigma$, given the voltage-to-current or Dirichlet-to-Neumann map on the boundary,

$$\Lambda_\sigma(f) = \left( \sigma \frac{\partial u}{\partial \nu} \right)\big|_{\partial \Omega},$$

where $\nu$ is the outer unit normal on $\partial \Omega$, and $u$ is the solution to (1) with $u|_{\partial \Omega} = f$. That is, knowing the current on the boundary $(\sigma \frac{\partial u}{\partial \nu})|_{\partial \Omega}$ induced by any given voltage potential $f$ on the boundary, find the conductivity $\sigma$ of the material. In his paper, Calderón considered a linearized version of this problem. The problem was then taken up by several authors. In [KV84], Kohn and Vogelius showed that the Dirichlet to Neumann map determines the conductivity on the boundary. The breakthrough came with the work [SU87], in which Sylvester and Uhlmann showed that a smooth conductivity can be recovered from knowledge of the map (2). The proof was based on the construction of the so-called complex geometrical optics solutions (CGO solutions), that are functions of the form

$$v(x) = e^{\xi \cdot x} \left( 1 + r(\xi, x) \right),$$

where $|\xi|$ is large and $r$ decreases in some sense as $|\xi| \to \infty$. Their method has since been adapted to show results for less regular conductivities, see e.g. [HT13, CR16], as well as for other equations, such as the time-harmonic Maxwell’s equations in electrodynamics [OS96, CZ14].

I am currently working on two related settings for the latter inverse problem. The framework is as follows: We consider the time-harmonic Maxwell’s equations with frequency $\omega$ for the electric field $E$ and magnetic field $H$ in a domain $\Omega \subset \mathbb{R}^3$,

$$\nabla \wedge E - i \omega \mu H = 0, \quad \nabla \wedge H + i \omega \gamma E = 0 \quad \text{in } \Omega,$$
with a tangential boundary condition of the form $\nu \wedge E|_{\partial \Omega} = f$. Here, $\nabla \wedge$ denotes the curl in $\mathbb{R}^3$; $\gamma = \varepsilon + i\sigma / \omega$; and the electric permittivity $\varepsilon$, the magnetic permeability $\mu$ and the conductivity $\sigma$ are nonnegative functions.

The direct problem is to find $E$ and $H$, given $\mu$, $\varepsilon$, and $\sigma$, and boundary data. The inverse problem consists in reconstructing the material parameters $\mu$, $\varepsilon$, and $\sigma$, given some knowledge of the behavior of $E$ and $H$ on the boundary. In the following sections, I will introduce the two situations that I am studying and describe my progress to date and next steps.

2 Inverse problem on a bounded Lipschitz domain with full boundary data.

We consider a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary. It is known that for a given set of Lipschitz continuous parameters $\mu$, $\varepsilon$, and $\sigma$, and tangential boundary data in a suitable space (here, $\text{Div} F$ denotes the surface divergence),

$$\nu \wedge E = f \in \mathcal{T} H_{\text{Div}}^{1/2}(\partial \Omega) = \{ F \in H^{1/2}(\partial \Omega)^3 \mid \nu \cdot F = 0, \text{Div} F \in H^{1/2}(\partial \Omega) \} \quad \text{on} \partial \Omega,$$

(5)

the equations [4] have a unique solution pair $(E, H)$ in the space $H_{\text{curl}}(\Omega) \times H_{\text{curl}}(\Omega)$, where

$$H_{\text{curl}}(\Omega) = \{ u \in L^2(\Omega)^3 : \nabla \wedge u \in L^2(\Omega) \}.$$  

(6)

For the inverse problem, for a given frequency $\omega$, suppose we know the so-called Cauchy data set

$$C(\mu, \varepsilon, \sigma; \omega) = \{ (\nu \wedge E|_{\partial \Omega}, \nu \wedge H|_{\partial \Omega}) : (E, H) \text{ solves } [4] \text{ with parameters } \mu, \varepsilon, \sigma \}.$$  

(7)

That is, we know the tangential components of any pair of functions $(E, H) \in H_{\text{curl}}(\Omega)^2$ that solve [4] almost everywhere in $\Omega$. The question is whether this set of data uniquely determines $\mu$, $\varepsilon$, and $\sigma$.

This problem has previously been treated in several instances given higher regularity of the domain and parameters; notably, in [OS96], uniqueness of the solution to the inverse problem has been shown for smooth parameters on a smooth domain; more recently, in [CZ14] the case of continuously differentiable parameters on a domain with $C^1$ boundary was examined. On the other hand, for the conductivity equation, uniqueness has been shown even for Lipschitz continuous parameters [CR16]; both [CR16] and [CZ14] made use of an averaged estimate that was first derived in [HT13], where the authors studied the inverse problem for the conductivity equation on a Lipschitz domain with $C^1$ conductivity. My aim is to prove the following result for the inverse problem for Maxwell’s equations.

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^3$ be a non-empty bounded domain such that $\partial \Omega$ is locally described by the graph of a Lipschitz function. Fix $\omega > 0$, and let $\mu_1, \varepsilon_1, \sigma_1, \mu_2, \varepsilon_2, \sigma_2 \in C^{0,1}(\Omega)$ be bounded Lipschitz functions such that for a positive constant $c_0$ and $j = 1, 2$,

$$0 < \mu_0 \leq \mu_j(x), \quad 0 < \varepsilon_0 \leq \varepsilon_j(x), \quad 0 \leq \sigma_j(x) \quad \forall x \in \overline{\Omega},$$

$$|\mu_j(x) - \mu_j(y)| \leq c_0|x - y|, \quad |\varepsilon_j(x) - \varepsilon_j(y)| \leq c_0|x - y|, \quad |\sigma_j(x) - \sigma_j(y)| \leq c_0|x - y| \quad \forall x, y \in \overline{\Omega}.$$

Assume further that $\mu_1(x) = \mu_2(x)$ and $\gamma_1(x) = \gamma_2(x)$ for all $x \in \partial \Omega$, where $\gamma_j(x) = \varepsilon_j(x) + i\sigma_j(x)/\omega$. Then

$$C_1 = C_2 \quad \Rightarrow \quad \mu_1 = \mu_2 \quad \text{and} \quad \gamma_1 = \gamma_2.$$

In the following I will describe in more detail the steps that I have taken towards the proof and outline the subsequent steps.

2.1 An auxiliary elliptic system

The first step in treating this problem is to augment Maxwell’s equations to obtain an elliptic system, following a procedure introduced in [OS96]. This is accomplished by adding two redundant equations and
rescaling the functions suitably. Assume $E$ and $H$ satisfy (4), and define the scalar potentials (which in this case vanish identically)

$$
\Phi = \frac{i}{\omega} \nabla \cdot (\gamma E), \quad \Psi = \frac{i}{\omega} \nabla \cdot (\mu H).
$$

These are used to modify Maxwell’s equations to obtain the equations

$$
\nabla \wedge E - \frac{1}{\gamma} \nabla \frac{1}{\mu} \Psi - i \omega \mu H = 0, \quad \nabla \wedge H + \frac{1}{\mu} \nabla \frac{1}{\gamma} \Phi + i \omega \gamma E = 0,
$$

which at the given level of regularity are understood in a weak sense. If we rescale the fields as

$$
e = \gamma^{1/2} E, \quad h = \mu^{1/2} H, \quad \phi = \frac{1}{\gamma \mu^{1/2}} \Phi, \quad \psi = \frac{1}{\gamma^{1/2} \mu} \Psi;
$$

from equations (8)- (9) it follows that the vector $X = (\phi, e, h, \psi)^T$ is a weak solution to the matrix differential equation

$$
P X := (P(i\nabla) - k + V) X = 0,
$$

where $P(i\nabla)$ is the elliptic first order matrix differential operator

$$
P(i\nabla) = i \begin{pmatrix}
0 & \nabla & 0 & 0 \\
\nabla & 0 & \nabla \wedge & 0 \\
0 & -\nabla \wedge & 0 & \nabla \\
0 & 0 & \nabla & 0
\end{pmatrix},
$$

and

$$
V = (k - \kappa) I_8 + (P(i\nabla) D) D^{-1},
$$

with $D = \text{diag}(\mu^{1/2}, \gamma^{1/2} I_3, \mu^{1/2} I_3, \gamma^{1/2})$, $k = \omega(\epsilon_0 \mu_0)^{1/2}$ and $\kappa = \omega(\mu \gamma)^{1/2}$. We also define the operator $P' := P(i\nabla) + k - V^T$, and note that

$$
(P(i\nabla) - k + V)(P(i\nabla) + k - V^T) = PP' = -(\Delta + k^2) + Q,
$$

a matrix Schrödinger operator, where $Q$ is the weakly defined matrix multiplier

$$
Q = VP(i\nabla) - P(i\nabla) V^T + k(V + V^T) - VV^T.
$$

Note that the system was modified in such a way that $Q$ turns out to be a zeroth order operator. Setting

$$
\alpha = \nabla \gamma / \gamma, \quad \beta = \nabla \mu / \mu, \quad Q \text{ has the following shape (in the following I will consistently write 8-vectors} \ w = (w_1, w_2, w_3, w_4)^T \text{ with scalar functions} \ w_1, w_4, \text{ and} \ w_2, w_3 \text{ being 3-vectors;} \ \varphi \text{ is a vector test function})
$$

$$
\langle Q w, \varphi \rangle = \int \left( \frac{1}{4} (|\alpha|^2 - 4\theta) [w_1 \varphi_1 + w_3 \cdot \varphi_3] - \frac{1}{2} \alpha \cdot \left[ \nabla (w_1 \varphi_1 - w_3 \cdot \varphi_3) + \nabla \cdot (w_3 \varphi_3^T + \varphi_3 w_3^T) \right] \right.
$$

$$
\hspace{1cm} + \frac{1}{4} (|\beta|^2 - 4\theta) [w_4 \varphi_4 + w_2 \cdot \varphi_2] - \frac{1}{2} \beta \cdot \left[ \nabla (w_4 \varphi_4 - w_2 \cdot \varphi_2) + \nabla \cdot (w_2 \varphi_2^T + \varphi_2 w_2^T) \right]
$$

$$
\hspace{1cm} - 2i\kappa \nabla \cdot \left[ w_1 \varphi_2 + w_2 \varphi_1 + w_3 \varphi_4 + w_4 \varphi_3 \right] \bigg),
$$

with $\theta = \omega^2 (\gamma \mu - \epsilon_0 \mu_0) = k^2 (\frac{\gamma \mu}{\epsilon_0 \mu_0} - 1)$. I will also be using the operator

$$
(P(i\nabla) + k - V^T)(P(i\nabla) - k + V) = PP' = -(\Delta + k^2) + \tilde{Q},
$$

where $\tilde{Q}$ is defined as

$$
\langle \tilde{Q} w, \varphi \rangle = \int \left( \frac{1}{4} (|\alpha|^2 - 4\theta) [w_1 \varphi_1 + w_3 \cdot \varphi_3] + \frac{1}{2} \beta \cdot \left[ \nabla (w_1 \varphi_1 - w_3 \cdot \varphi_3) + \nabla \cdot (w_3 \varphi_3^T + \varphi_3 w_3^T) \right] \right.
$$

$$
\hspace{1cm} + \frac{1}{4} (|\beta|^2 - 4\theta) [w_4 \varphi_4 + w_2 \cdot \varphi_2] + \frac{1}{2} \alpha \cdot \left[ \nabla (w_4 \varphi_4 - w_2 \cdot \varphi_2) + \nabla \cdot (w_2 \varphi_2^T + \varphi_2 w_2^T) \right]
$$

$$
\hspace{1cm} + 2i\kappa \nabla \cdot \left[ w_3 \wedge \varphi_2 - w_2 \wedge \varphi_3 \right] \bigg),
$$

(11)
It is noteworthy that the first and last components of this operator decouple from the rest and allow to treat those separately. This will be important later on for finding solutions that have vanishing first and last components, since in view of (8), only those will yield solutions to Maxwell’s equations.

The following propositions establish how solutions to Schrödinger equations yield solutions to the auxiliary equations

\[ P' v = (P(i\nabla) + k - V^T)v = 0 \]

**Proposition 2.2.** If \( w \in H^1(\Omega)^8 \) solves the vector Schrödinger equation \( [-(\Delta + k^2) + Q]w = 0 \) weakly in \( \Omega \), i.e., \( w \) satisfies

\[
\int_{\mathbb{R}^3} \sum_{j=1}^8 (\nabla w_j \cdot \nabla \varphi_j) - k^2 w \cdot \varphi \, dx + \langle Qw, \varphi \rangle = 0
\]

for all \( \varphi \in C_c^\infty(\Omega)^8 \), then \( v = P'w \) is a weak solution to

\[ P' v = 0 \quad \text{in} \quad \Omega, \]

and \( v \in H^1_{\text{loc}}(\Omega)^8 \).

**Proof.** In order to show that \( v \) is a weak solution to (14), we need to show that

\[
0 = \int_{\mathbb{R}^3} v \cdot P' \varphi \, dx = \int_{\mathbb{R}^3} P'w \cdot P' \varphi \, dx
\]

\[
= \int_{\mathbb{R}^3} P(i\nabla)w \cdot P(i\nabla)\varphi + k^2 w \cdot \varphi + kw \cdot P(i\nabla)\varphi + P(i\nabla)w \cdot k\varphi
\]

\[- V^T w \cdot P(i\nabla)\varphi - P(i\nabla)w \cdot V^T \varphi - kw \cdot V^T \varphi - V^T w \cdot k\varphi + V^T w \cdot V^T \varphi \, dx
\]

for all \( \varphi \in C_c^\infty(\Omega)^8 \), and we will do so by showing that this integral equals the left-hand side of (13). We first note that

\[
\int_{\mathbb{R}^3} P(i\nabla)w \cdot P(i\nabla)\varphi \, dx = -\int_{\mathbb{R}^3} \sum_{l=1}^8 \nabla w_l \cdot \nabla \varphi_l \, dx.
\]

The third and fourth terms cancel after an integration by parts, and it is straightforward to check that the last five terms give \( -\langle Qw, \varphi \rangle \).

In order to see that \( v \in H^1_{\text{loc}}(\Omega)^8 \), we first note that \( v \in L^2(\Omega)^8 \); hence, by (14), \( P(i\nabla)v \in L^2(\Omega)^8 \), and the claim follows by elliptic regularity. \( \square \)

The following analog holds when switching the roles of \( P \) and \( P' \):

**Proposition 2.3.** If \( w \in H^1(\Omega)^8 \) is a weak solution to \( [-(\Delta + k^2) + \tilde{Q}]w = 0 \) in \( \Omega \), i.e., \( w \) satisfies

\[
\int_{\mathbb{R}^3} \sum_{j=1}^8 (\nabla w_j \cdot \nabla \varphi_j) - k^2 w \cdot \varphi \, dx + \langle \tilde{Q}w, \varphi \rangle = 0
\]

for all \( \varphi \in C_c^\infty(\Omega)^8 \), then \( v = Pw \) is a weak solution to

\[ P' v = 0 \quad \text{in} \quad \Omega, \]

and \( v \in H^1_{\text{loc}}(\Omega)^8 \). \( \square \)
2.2 Integral formula

The next step is to obtain a certain integral identity involving the unknown parameters as well as solutions to the auxiliary equations, which will be the starting point of the uniqueness proof. So suppose that we have two sets of parameters, \( \mu_j, \varepsilon_j, \sigma_j \in C^{0,1}(\Omega) \) that have the same Cauchy set on \( \partial \Omega \), and such that on \( \partial \Omega \), \( \mu_1 = \mu_2, \varepsilon_1 = \varepsilon_2 \) and \( \sigma_1 = \sigma_2 \). Then we can perform a Whitney extension of the parameters to obtain Lipschitz continuous functions on the whole space such that \( \mu_1 = \mu_2, \varepsilon_1 = \varepsilon_2 \) and \( \sigma_1 = \sigma_2 \) outside \( \Omega \), and the Lipschitz constant on \( \mathbb{R}^3 \) depends only on the Lipschitz constant \( c_0 \) on \( \Omega \) and the space dimension, cf. [Ste70, Section VI.2]. We take an extension such that for some \( R > 0 \) sufficiently large such that \( \Omega \subset B(0, R) \),

\[
\mu_j(x) \equiv \mu_o, \quad \gamma_j(x) \equiv \varepsilon_o, \quad x \in B(0, R)^c.
\]

Let \( Q_j \) denote the weak potential with parameters \( \mu_j, \gamma_j, j = 1, 2 \). We also denote

\[
P_j = P(i\nabla) - k + V(\mu_j, \gamma_j), \quad P_j' = P(i\nabla) + k - V(\mu_j, \gamma_j).
\]

**Proposition 2.4.** Let \( w_1 \in H^{1}_{loc}(\mathbb{R}^3)^8 \) be a weak solution to the equation \( [-\Delta + k^2] w_1 = 0 \) in \( \Omega \), i.e., \( w \) satisfies (13) with \( Q = Q_1 \) for all \( \varphi \in C^{\infty}_c(\Omega)^8 \), and assume that \( v_1 = P_1' w_1 \) has vanishing first and last components. Furthermore, let \( v_2 \in H^{1}_{loc}(\mathbb{R}^3)^8 \) satisfy

\[
P_2' v_2 = 0 \quad \text{in} \quad \Omega.
\]

Then if the Cauchy data sets \( C_1 \) and \( C_2 \) are equal, the following integral identity holds:

\[
(\langle Q_2 - Q_1 \rangle w_1, v_2) = 0.
\]

**Proof.** As in the proof of Proposition 2.2 we see that for \( l = 1, 2 \),

\[
\int_{\mathbb{R}^3} \sum_{j=1}^9 (\nabla w_{1,j} \cdot \nabla \varphi_j) - k^2 w_1 \cdot \varphi \, dx + \langle Q_l w_1, \varphi \rangle = \int_{\mathbb{R}^3} P_l^i w_1 \cdot P_l^i \varphi \, dx
\]

for all \( \varphi \in C^{\infty}_c(\mathbb{R}^3)^8 \); by a density argument, the same holds for \( \varphi \in H^{1}_{loc}(\mathbb{R}^3)^8 \), so we may let \( \varphi = v_2 \). Subtracting the equation for \( l = 1 \) from that for \( l = 2 \), and using the fact that the two sets of parameters agree outside \( \Omega \), we get

\[
(\langle Q_2 - Q_1 \rangle w_1, v_2) = \int_{\Omega} P_2^i w_1 \cdot P_2^i v_2 \, dx - \int_{\Omega} P_1^i w_1 \cdot P_1^i v_2 \, dx = - \int_{\Omega} v_1 \cdot P_1^i v_2 \, dx,
\]

by the definition of \( v_1 \) and the assumption on \( v_2 \), by which the first integral vanishes. By construction, the assumption that \( v_{1,1} = v_{1,4} = 0 \) guarantees that \( u = (0, E, H, 0) := (0, \gamma_1^{-1/2} v_{1,2}, \mu_1^{-1/2} v_{1,3}, 0) \) is a solution to Maxwell’s equations in \( \Omega \) with parameters \( \mu_1 \) and \( \gamma_1 \),

\[
\nabla \times E - i \omega \mu_1 H = 0, \quad \nabla \times H + i \omega \gamma_1 E = 0.
\]

Thus, integrating by parts and using the fact that \( P_1 v_1 = 0 \) weakly, we obtain

\[
- \int_{\Omega} v_1 \cdot P_1^i v_2 \, dx = \int_{\partial \Omega} (v_{1,2} \cdot \nu) v_{2,1} + v_{1,2} \cdot (\nu \times v_{2,3}) - v_{1,3} \cdot (\nu \times v_{2,2}) + (v_{1,3} \cdot \nu) v_{2,4} \, dS
\]

\[
= \int_{\partial \Omega} (\gamma_1 E \cdot \nu) \gamma_1^{-1/2} v_{2,1} - (\nu \times E) \cdot \gamma_1^{1/2} v_{2,3} + (\nu \times H) \cdot \mu_1^{-1/2} v_{2,2} + (\mu_1 H \cdot \nu) \mu_1^{1/2} v_{2,4} \, dS.
\]

(21)

Now we use the fact that \( (\nu \times E) |_{\partial \Omega}, (\nu \times H) |_{\partial \Omega} \) \( \in C_1 = C_2 \). This gives the existence of a solution \( (\tilde{E}, \tilde{H}) \in H_{curl}(\Omega)^2 \) to Maxwell’s equations with parameters \( \mu_2, \gamma_2 \) in \( \Omega \),

\[
\nabla \times \tilde{E} - i \omega \mu_2 \tilde{H} = 0, \quad \nabla \times \tilde{H} + i \omega \gamma_2 \tilde{E} = 0,
\]

(22)
with the same Cauchy data. We set \( g = (0, \gamma_2^{1/2} \vec{E}, \mu_2^{1/2} \vec{H}, 0) \) almost everywhere in \( \Omega \), and \( g = v_1 \) almost everywhere outside of \( \Omega \). Then by (17), we have \( \int_{\Omega} g \cdot \mathcal{P}_2 v_2 dx = 0 \), and an analogous calculation to that above shows

\[
0 = \int_{\Omega} g \cdot \mathcal{P}_2 v_2 dx
= -\int_{\partial \Omega} (\gamma_2 \vec{E} \cdot \nu) \gamma_2^{-\frac{1}{2}} v_{2,1} - (\nu \wedge \vec{E}) \cdot \gamma_2^\frac{1}{2} v_{2,3} + (\nu \wedge \vec{H}) \cdot \mu_2^\frac{1}{2} v_{2,2} + (\mu_2 \vec{H} \cdot \nu) \mu_2^{-\frac{1}{2}} v_{2,4} dS. \tag{23}
\]

Adding (23) to (21), and using the fact that \( \mu_1 = \mu_2 \) and \( \gamma_1 = \gamma_2 \) on \( \partial \Omega \), we obtain

\[
\langle (Q_2 - Q_1)w_1, v_2 \rangle = \int_{\partial \Omega} (\gamma_1 \vec{E} \cdot \nu) \gamma_1^{-\frac{1}{2}} v_{2,1} - (\gamma_2 \vec{E} \cdot \nu) \gamma_2^{-\frac{1}{2}} v_{2,1} - (\mu_2 \vec{H} \cdot \nu) \mu_2^{-\frac{1}{2}} v_{2,4} dS
= \frac{1}{i\omega} \int_{\partial \Omega} \{[\nabla \wedge (H - \vec{H})] \cdot \nu\} \gamma_1^{-\frac{1}{2}} v_{2,1} - \{[\nabla \wedge (E - \vec{E})] \cdot \nu\} \mu_1^{-\frac{1}{2}} v_{2,4} dS,
\]

where the last equality was obtained using Maxwell’s equations (20) and (22). Now we use the boundary identity \( \nu \cdot \nabla \wedge f = -\text{Div} \cdot (\nu \wedge f) \) in \( H^{-1/2}(\partial \Omega) \), cf. [Mon03], which holds for \( f \in H_{\text{curl}}(\Omega) \), for \( H - \vec{H} \) as well as \( E - \vec{E} \), to obtain

\[
\langle (Q_2 - Q_1)w_1, v_2 \rangle = -\frac{1}{i\omega} \int_{\partial \Omega} \{\text{Div} \cdot [\nu \wedge (H - \vec{H})]\} \gamma_1^{-\frac{1}{2}} v_{2,1} - \{\text{Div} \cdot [\nu \wedge (E - \vec{E})]\} \mu_1^{-\frac{1}{2}} v_{2,4} dS,
\]

and now this integral vanishes by our choice of \( \vec{E} \) and \( \vec{H} \), which finishes the proof.

The crucial step in the proof of Theorem 2.1 is to construct a sufficient number of CGO solutions \( w_1 \) to the auxiliary Schrödinger equation (that in turn yield solutions to Maxwell’s equations) and \( v_2 \) to the auxiliary first-order equation such that from the integral equation (18) we will be able to deduce certain relations between the parameters that allow us to conclude that in fact \( (\mu_2, \varepsilon_2, \sigma_2) = (\mu_1, \varepsilon_1, \sigma_1) \).

In order to show existence of these solutions, an a priori estimate on suitable function spaces is needed. I am using a generalization of an a estimate derived in [CR16].

### 2.3 A priori estimate

We start by introducing the spaces the solutions will be constructed in, as well as some auxiliary norms. The following are analogs of spaces introduced in [HT13], for vector-valued functions.

For \( \zeta \in \mathbb{C}^3 \), define the polynomial \( p_\zeta(x) = |x|^2 - 2i\zeta \cdot x \). For \( b \in \mathbb{R} \) define \( X^b_\zeta \) by

\[
\|w\|_{X^b_\zeta} = \left( \sum_{j=1}^{8} \left\| p_\zeta^b \hat{w}_j \right\|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} < \infty,
\]

and \( X^b_\zeta \) by

\[
\|w\|_{X^b_\zeta} = \left( \sum_{j=1}^{8} \left\| |\zeta| + |p_\zeta| \right\|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} < \infty,
\]

where \( \hat{w} \) denotes the Fourier transform of \( w \). Furthermore, following [CR16], for \( M, \tau > 0 \), define the Fourier multiplier \( m \) by

\[
m(\xi) = \left( M^{-1} |\xi|^2 - \tau^2 \right)^2 + M^{-1} \tau^2 |\xi|^2 + M\tau^2 \right)^{1/2},
\]

and for \( u \in \mathcal{S}(\mathbb{R}^3) \), the Schwartz class of rapidly decaying functions, we define the norm for \( b \in \mathbb{R} \),

\[
\|u\|_{Y^b} = \|m^b \hat{u}\|_{L^2}.
\]
For vector-valued functions, \(u \in \mathcal{S}(\mathbb{R}^3)^8\), we define the norm as usual by
\[
\|u\|_{Y^s}^2 = \sum_{k=1}^{8} \|u_k\|_{Y^s}^2.
\]

The following estimate was proved (for general dimension \(n\)) in \[\text{CR16}, \text{Lemma 2.3}\]:

**Lemma 2.5.** Let \(\sigma\) be a Lipschitz continuous function that is constant outside a set of compact support, and let \(A > 1\) be such that
\[
\|\sigma^{-1} \nabla \sigma\|_{L^\infty} < A.
\]
Define \(q = \sigma^{-1/2} \Delta \sigma^{1/2}\) in the weak sense, that is, for \(\phi, \psi \in H^1_{\text{loc}}(\mathbb{R}^3)\)
\[
\langle q \phi, \psi \rangle = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \log \sigma|^2 \phi \psi \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \nabla \log \sigma \cdot \nabla (\phi \psi) \, dx,
\]
and furthermore, for a rotation \(T\),
\[
\langle T^* q \phi, \psi \rangle = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \log (T x)|^2 \phi(x) \psi(x) \, dx - \frac{1}{2} \int_{\mathbb{R}^3} \nabla \log (T x) \cdot \nabla (\phi(x) \psi(x)) \, dx.
\]
There is a constant \(C\) such that for \(M = CR^2 A^4\), \(\tau > 8MR\), and any \(u \in \mathcal{S}(\mathbb{R}^3)\) with \(\text{supp } u \subset \{|x_3| < R\}\),
\[
\|u\|_{Y^{-1/2}} \lesssim \|(-\Delta + 2\tau \partial_{x_3} - \tau^2 + T^* q) u\|_{Y^{-1/2}}.
\]
The implicit constant depends on \(A\) and \(R\).

The following vector-valued generalization of this estimate with the weak matrix multiplier \(Q\) of the form \[\text{(10)}\] (or \(\tilde{Q}\) as in \[\text{(11)}\]) holds: Let \(A > 1\) be such that
\[
\max \{\|\alpha\|_{L^\infty}, \|\beta\|_{L^\infty}\} < A,
\]
and let \(Q\) be as in \[\text{(10)}\] (or \[\text{(11)}\]). Then, as in the scalar case, there is a constant \(C\) such that for \(M = CR^2 A^4\), \(\tau > 8MR\), and any \(u \in \mathcal{S}(\mathbb{R}^3)^8\) with \(\text{supp } u \subset \{|x_3| < R\}\),
\[
\|u\|_{Y^{-1/2}} \lesssim \|(-\Delta + 2\tau \partial_{x_3} - \tau^2 + T^* Q) u\|_{Y^{-1/2}}.
\]
I will use this to establish the following estimate in the \(X^s_T\) norms, which is analogous to Proposition 2.4 in \[\text{CR16}\].

**Proposition 2.6.** Let \(\zeta \in \mathbb{C}^3\) such that \(\text{Re} \, \zeta \perp \text{Im} \, \zeta\) and \(|\text{Re} \, \zeta|^2 = \tau^2 = |\text{Im} \, \zeta|^2 - k^2\). Furthermore, fix a constant \(A > \max\{\|\alpha\|_{L^\infty}, \|\beta\|_{L^\infty}, 1\}\). Then there exists an absolute constant \(C\) such that for \(\tau > CR^3 A^4\),
\[
\|u\|_{X^{1/2}_T} \lesssim \|(-\Delta \zeta \cdot \nabla + Q) u\|_{X^{-1/2} T},
\]
provided that \(u \in \mathcal{S}(\mathbb{R}^3)^8\) with \(\text{supp } u \subset \{|x| < R\}\). The implicit constant depends on \(A\) and \(R\).

**Proof.** Let \(T\) be a rotation such that \(\text{Re} \, \zeta = \tau e_3\), where \(e_3 = (0, 0, 1)^T\). Let \(w \in \mathcal{S}(\mathbb{R}^3)^8\) with \(\text{supp } w \subset \{|x| < R\}\) and consider \(v(x) = T^* w(x) = w(T x)\). By Lemma 2.5, we have that for \(M = CR^2 A^4\) and \(\tau > 8MR\),
\[
\|v\|_{Y^{-1/2}} \lesssim \|(-\Delta + 2\tau \partial_{x_3} - \tau^2 + T^* Q) v\|_{Y^{-1/2}}.
\]
The right-hand function is
\[
(-\Delta + 2\tau \partial_{x_3} - \tau^2 + T^* Q) w(T x) = -T^* (\Delta w) + T^* 2(\text{Re} \, \zeta \cdot \nabla) I_8 w - T^* |\text{Re} \, \zeta|^2 I_8 w + T^* Q w.
\]
Writing \( w(x) = e^{-i \text{Im} \cdot \xi \cdot x} u(x) \), with \( u \in \mathcal{S}(\mathbb{R}^3)^8 \) with supp \( u \subset \{ |x| \leq R \} \), one can easily check that
\[
T^*[\Delta + 2(\text{Re} \xi \cdot \nabla) - |\text{Re} \xi|^2 + Q]w = T^*[e^{-i \text{Im} \cdot \xi \cdot x}(-\Delta + k^2 + 2\xi \cdot \nabla + Q)u],
\]
and thus (25) yields
\[
\|T^*[e^{-i \text{Im} \cdot \xi \cdot x}u]\|_{Y^{1/2}} \lesssim \|T^*[e^{-i \text{Im} \cdot \xi \cdot x}(-\Delta + k^2 + 2\xi \cdot \nabla + Q)u]\|_{Y^{-1/2}}.
\] (26)

In order to compute these norms and relate them to the \( X_\xi^b \) norms, we need the Fourier transform of a function of the form \( T^*[e^{-i \text{Im} \cdot \xi \cdot x}f] \), for \( f \in X_\xi^b \). A quick computation shows that
\[
\mathcal{F}[T^*[e^{-i \text{Im} \cdot \xi \cdot x}f]](\xi) = \mathcal{F}[e^{-i \text{Im} \cdot \xi \cdot x}f](T\xi) = \mathcal{F}[f](T\xi + \text{Im} \xi),
\]
thus,
\[
\|T^*[e^{-i \text{Im} \cdot \xi \cdot x}f]\|_{Y^b}^2 = \int_{\mathbb{R}^3} m(\xi)^{2b} |\hat{f}(T\xi + \text{Im} \xi)|^2 d\xi = \int_{\mathbb{R}^3} m(T^{-1}(\xi - \text{Im} \xi))^{2b} |\hat{f}(\xi)|^2 d\xi
\]
after change of variables. The definition of the multiplier shows that \( m(T^{-1}(\xi - \text{Im} \xi))^{2b} \sim (|\xi| + |p_\xi(\xi)|)^{2b} \), so that the right hand integral is proportional to \( \|f\|_{X_\xi^b}^2 \), and using this on both sides of (26) gives the estimate
\[
\|u\|_{X_\xi^{1/2}} \lesssim \|(-\Delta + k^2 + 2\xi \cdot \nabla + Q)u\|_{X_\xi^{-1/2}}.
\]

Applying the triangle inequality on the right-hand side and using the definition of the norms, we obtain
\[
\|u\|_{X_\xi^{1/2}} \lesssim \|(-\Delta + 2\xi \cdot \nabla + Q) u\|_{X_\xi^{-1/2}} + k^2 |\xi|^{-2} \|u\|_{X_\xi^{1/2}},
\]
and now the last term on the right can be absorbed into the left-hand side if \( |\xi| \) is large enough, which finishes the proof. \( \square \)

This estimate will be used to construct the solutions needed in the integral formula (18). Existence of such solutions can be shown employing the Riesz representation theorem. In this step it is crucial that the solutions \( w_1 \) are such that \( v_1 = \mathcal{P}_I' w_1 \) has vanishing first and last components; only functions \( w_1 \) with this property actually yield solutions to the original Maxwell system under consideration. To facilitate this, a uniqueness result for the CGO solutions is needed. At this point, I am studying a method in [NS10], where uniquely determined solutions to the conductivity equation with certain partial data were used in finding a reconstruction method for the material parameter. The idea is to consider a certain orthogonality property in the solution space and show that there is a unique solution with this property. The choice of condition to impose is informed by methods used in the existence proof.

The final step in the proof of uniqueness of the parameters then is to plug the CGO solutions into the integral formula and take the limit of the large parameter \(|\xi| \to \infty\), using an averaged estimate derived in [HT13] to control certain terms. The result will be a set of homogeneous differential equations for the differences of the parameters, \( \mu_1 - \mu_2 \) and \( \gamma_1 - \gamma_2 \). Then a unique solvability result for these equations will yield \( \mu_1 = \mu_2 \) and \( \gamma_1 = \gamma_2 \) and finish the proof.

### 3 Inverse problem on an unbounded domain with partial data.

The second problem that I have been studying is a partial data problem, that is, one in which boundary data is available only on a subset of the boundary. Such problems arise naturally in applications where it is not possible to take measurements on the whole surface of a given body, for example when part of the boundary is not accessible, or when it would simply be too costly to do so. The setting for the problem that I am considering is that of a slab, an infinite domain bounded by two parallel planes:
\[
\Omega = \{ x \in \mathbb{R}^3 : 0 < x_3 < L \}. 
\]
The unboundedness of the domain necessitates imposing additional conditions on the solutions concerning their behavior at infinity in order to guarantee unique solvability of the equations: a suitable radiation condition is needed.

The treatment of the partial data case follows the same basic idea as used in the full data case: obtain an integral formula containing the parameters as well as special solutions, then construct suitable CGO solutions to conclude uniqueness of the parameters from that integral formula. The procedure is complicated by the fact that in deriving the integral formula, certain boundary integrals with unknown functions appear, and estimates are needed to control these boundary terms.

There are a number of results on partial data problems in a slab for Schrödinger equations: In [LU10], the authors study the scalar Schrödinger equation in the two cases that partial Dirichlet and Neumann data are given (i) on the same boundary hyperplane or (ii) on opposite boundary hyperplanes. The authors use a Carleman estimate to show that the boundary integrals are negligible in case (i); in case (ii), special CGO solutions that vanish on one boundary hyperplane are constructed using a reflection argument. In [KL12], a Schrödinger operator with a magnetic term was studied assuming the same geometry, using reflection arguments to construct the necessary CGOs; in [Li12a, Li12b] a matrix Schrödinger operator was studied in each of these cases.

The setting for the problem that I am considering is as follows: Let \( \Omega = \{ x \in \mathbb{R}^3 : 0 < x_3 < L \} \), and denote the boundary hyperplanes \( \Gamma_1 = \{ x_3 = L \} \) and \( \Gamma_2 = \{ x_3 = 0 \} \); furthermore, let \( \Gamma_1' \subset \Gamma_1 \) and \( \Gamma_2' \subset \Gamma_2 \) be open subsets of the boundary planes. We consider the time-harmonic Maxwell equations

\[
\nabla \wedge E - i \omega \mu H = 0, \quad \nabla \wedge H + i \omega \gamma E = 0 \quad \text{in} \ \Omega,
\]

suitable radiation condition for \( E \) and \( H \) as \( |(x_1, x_2)| \to \infty \),

(27)

where \( \mu \) and \( \gamma = \varepsilon + i \sigma / \omega \) are functions such that for some radius \( R > 0 \), \( \mu \equiv \mu_0 > 0 \), \( \varepsilon \equiv \varepsilon_0 > 0 \) and \( \sigma \equiv 0 \) outside \( B(0, R) \); and some partial Cauchy data. We want to investigate cases analogous to those considered in [LU10]: on the one hand, knowledge of the tangential boundary components \( \nu \wedge E \) and \( \nu \wedge H \) on (parts of) opposite boundary hyperplanes, and on the other hand, knowledge of the boundary values on the same boundary hyperplane. Consequently, the partial Cauchy data set will be of the form

\[
C(\mu, \varepsilon, \sigma; \omega) = \{((\nu \wedge E)|_{\Gamma_1}, (\nu \wedge H)|_{\Gamma_2}) : (E, H) \text{ solves } (27)\}
\]

(28)

in the first case, and in the latter case

\[
C(\mu, \varepsilon, \sigma; \omega) = \{((\nu \wedge E)|_{\Gamma_1}, (\nu \wedge H)|_{\Gamma_2}) : (E, H) \text{ solves } (27)\}.
\]

(29)

The question I am concerning myself with is whether knowledge of the Cauchy data set in either case is sufficient to uniquely determine the parameters \( \mu \) and \( \gamma \) in \( \Omega \).

In the treatment of this problem, the first step is to show the well-posedness of the direct problem, which to the best of my knowledge hasn’t been studied before. The direct problem is the following: Given a fixed frequency \( \omega > 0 \), twice continuously differentiable functions \( \mu, \varepsilon, \sigma \) such that for some \( R > 0 \),

\[
\mu(x) \equiv \mu_0 > 0, \ \varepsilon(x) \equiv \varepsilon_0 > 0, \ \sigma(x) = 0, \ \ x \in B(0, R)^c,
\]

as well as a tangential boundary datum \( f \in \mathcal{TH}^{3/2}(\Gamma_1) = \{ F \in H^{3/2}(\Gamma_1)^3 \ | \ \nu \cdot F = 0 \} \), find vector functions \( E \) and \( H \) that solve

\[
\nabla \wedge E(x) - i \omega \mu(x) H(x) = 0 \quad \text{in} \ \Omega, \quad \nu \wedge E|_{\Gamma_1} = f,
\]

\[
\nabla \wedge H(x) + i \omega \gamma(x) E(x) = 0 \quad \text{in} \ \Omega, \quad \nu \wedge E|_{\Gamma_2} = 0,
\]

radiation condition for \( E \) and \( H \) as \( |(x_1, x_2)| \to \infty \).

(30c)

I will postpone the discussion of the necessary radiation condition for now and first break the problem into more workable pieces. The following is motivated by the Lax-Phillips method used in showing well-posedness for Schrödinger equations in [LU10, Li12b], splitting the problem into one with constant coefficients in the whole slab and one with non-constant coefficients in a bounded subdomain of the slab. I am following along these lines to show unique solvability of the direct problem for Maxwell’s equations formulated above.
3.1 Well-posedness of the direct problem

We can first simplify the treatment by reducing the problem to one with zero boundary condition: let 
\( E_o \in H^2(\mathbb{R}^3)^3 \) be a compactly supported function such that \( \nu \wedge E_o = f \) on \( \Gamma_1 \) and \( \nu \wedge E_o = 0 \) on \( \Gamma_2 \), and look for a pair of solutions \((\tilde{E}, \tilde{H}) = (E + E_o, H)\), where \( E \) and \( H \) solve

\[
\nabla \wedge E(x) - i \omega \mu(x) H(x) = -\nabla \wedge E_o(x) =: F_1(x) \quad \text{in } \Omega, \quad \nabla \wedge H(x) + i \omega \gamma(x) E(x) = -i \omega \gamma(x) E_o(x) =: F_2(x) \quad \text{in } \Omega, \quad \nu \wedge E|_{\partial \Omega} = 0, \tag{31}\n\]
as well as a radiation condition as \(|(x_1, x_2)| \to \infty\) for \( E \) and \( H \). Note that, in particular, \( F_1, F_2 \in H^1(\mathbb{R}^3)^3 \), and \( F_1, F_2 \) are compactly supported.

Next, we construct a bounded subdomain \( \Omega_b \subset \Omega \) as follows: fix \( R > 0 \) such that \( \mu \) and \( \varepsilon \) are constant and \( \sigma \) vanishes outside the ball \( B(0, R) \). We then inscribe a torus of minor radius \( L/2 \) and major radius \( R'' > R \) centered at the \( z \) axis into \( \Omega \) and let \( \Omega_b \) be the convex hull of this torus; in this construction, we choose \( R'' \) such that \( \omega \) is not an eigenvalue for the Maxwell system on \( \Omega_b \). Note that \( \Omega_b \) has \( C^{1,1} \) boundary.

Now we pick a function \( \varphi \in C^\infty(\mathbb{R}^3) \) with \( \varphi(x) = 1 \) for all \( x \in B(0, R') \) and \( \varphi(x) = 0 \) for all \( x \in B(0, R')^c \) for some \( R < R' < R'' \). The goal is to obtain a pair of solutions to (31) of the form

\[
E = E_1 - \varphi(E_1 - E_2), \quad H = H_1 - \varphi(H_1 - H_2), \tag{32}\n\]
where \((E_1, H_1)\) solves the problem with constant coefficients in the whole domain \( \Omega \),

\[
\nabla \wedge E_1 - i \omega \mu_o H_1 = \tilde{F}_1, \quad \nabla \wedge H_1 + i \omega \varepsilon_o E_1 = \tilde{F}_2 \quad \text{in } \Omega, \quad \nabla \wedge E_1 = 0 \quad \text{on } \partial \Omega, \quad \nu \wedge E_1 = \nu \wedge E_2 \quad \text{on } \partial \Omega_b, \tag{33b,c,d}\n\]

and \((E_2, H_2)\) solves the problem with non-constant coefficients in \( \Omega_b \),

\[
\nabla \wedge E_2 - i \omega \mu H_2 = \tilde{F}_1, \quad \nabla \wedge H_2 + i \omega \varepsilon E_2 = \tilde{F}_2 \quad \text{in } \Omega_b, \quad \nu \wedge E_2 = \nu \wedge E_1 \quad \text{on } \partial \Omega_b, \tag{34b,c,d}\n\]

where \( \tilde{F}_1, \tilde{F}_2 \in H^1(\mathbb{R}^3)^3 \) are compactly supported functions that will be determined in the following. Plugging the ansatz (32) into Maxwell’s equations, the equations are clearly satisfied inside \( \Omega \cap B(0, R) \), since \((E_2, H_2)\) solve (34). In the region \( \Omega_b \setminus B(0, R) \) we obtain

\[
\nabla \wedge E - i \omega \mu_o H = \tilde{F}_1 - \nabla \varphi \wedge (E_1 - E_2), \quad \nabla \wedge H + i \omega \varepsilon_o E = \tilde{F}_2 - \nabla \varphi \wedge (H_1 - H_2), \n\]

so in order for \((E, H)\) to satisfy (31), \( \tilde{F} = (\tilde{F}_1, \tilde{F}_2) \) needs to satisfy

\[
(I + K) \tilde{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad \text{in } \Omega_b \setminus B(0, R), \tag{35}\n\]

where

\[
K \tilde{F} = \begin{pmatrix} \nabla \varphi \wedge (E_2 - E_1) \\ \nabla \varphi \wedge (H_2 - H_1) \end{pmatrix}, \quad \text{in } \Omega_b \setminus B(0, R). \tag{36}\n\]

with \((E_1, H_1)\) being the solution to (33) and \((E_2, H_2)\) the solution to (34); we define \( K \tilde{F} = 0 \) in \( \Omega \cap B(0, R) \). In order to obtain the right-hand functions \( \tilde{F}_1, \tilde{F}_2 \), we need to show that (35) is uniquely solvable. We postpone this and first concern ourselves with the solvability of each of the systems (33) and (34); we begin with the constant coefficient system (33).
System with constant coefficients on the slab. In this case we can transform Maxwell’s equations to obtain a vector Helmholtz equation: Taking the divergence of the second equation, we obtain the identity
\[ \nabla \cdot E_1 = \frac{1}{i\omega \varepsilon_0} \nabla \cdot \tilde{F}_2. \] (37)
Taking the curl of the first equation and then using the second equation to substitute \( \nabla \wedge H_1 \), as well as the identity \( \nabla \wedge \nabla = -\Delta + \nabla \nabla \cdot \) and (37), we arrive at
\[ (-\Delta - k^2)E_1 = i\omega \mu_0 \tilde{F}_2 + \nabla \wedge \tilde{F}_1 - \frac{1}{i\omega \varepsilon_0} \nabla \nabla \cdot \tilde{F}_2 =: G, \] (38)
with \( k = \omega^2 \mu_0 \varepsilon_0 \); the function \( G \in H^{-1}(\mathbb{R}^3)^3 \) is compactly supported. Since on \( \partial \Omega \), \( \nu = \pm e_3 \), the boundary condition gives that \( E_{1,1} = E_{1,2} = 0 \) on \( \partial \Omega \). So we can expand \( E_{1,1} \) and \( E_{1,2} \) as sine series in terms of \( x_3 \),
\[ E_{1,j}(x) = \sum_{m=1}^{\infty} E^j_m(x_1, x_2) \sin \left( \frac{m\pi x_3}{L} \right). \]
The radiation condition for \( E_{1,1} \) and \( E_{1,2} \) concerns the coefficients of this expansion: in order to guarantee a unique solution to the Helmholtz equation on the slab, the coefficients have to satisfy the following conditions:
We need to require that \( k \) is such that \( k \neq m\pi/L \) for all \( m \in \mathbb{N} \). Then,

(i) for \( m \) such that \( k^2 - m^2 \pi^2/L^2 < 0 \), \( E^m_m \in H^{1}_{\text{loc}}(\mathbb{R}^2) \);

(ii) for \( m \) such that \( k^2 - m^2 \pi^2/L^2 > 0 \), set \( k_m = \sqrt{k^2 - m^2 \pi^2/L^2} \), then \( E^m_m \) satisfies, with \( x' = (x_1, x_2) \),
\[ E^m_m(x') = O(r^{-1/2}), \quad \left( \frac{\partial}{\partial r} - ik_m \right) E^j_m = o(r^{-1/2}), \quad r = |x'| \to \infty. \]
The latter is a Sommerfeld radiation condition for \( E^j_m \); this so-called partial radiation condition was introduced in [Sve50]. In order to obtain the unique solutions with these properties, we use the fundamental solution for the Helmholtz equation on the slab with homogeneous Dirichlet boundary condition,
\[ \Phi(x, y) = \sum_{m=1}^{\infty} -\frac{1}{2L} \sin \left( \frac{m\pi x_3}{L} \right) \sin \left( \frac{m\pi y_3}{L} \right) H^1_0(k_m|x'| - |y'|), \] (39)
where \( k_m = \sqrt{k^2 - m^2 \pi^2/L^2} \), and \( H^1_0 \) is the Hankel function of first kind. We obtain the solutions in \( H^1(\Omega) \)
\[ E_{1,1}(x) = \int_\Omega \Phi(x, y) G_1(y) dy, \quad E_{1,2}(x) = \int_\Omega \Phi(x, y) G_2(y) dy. \] (40)
We need a different approach for \( E_{1,3} \), however, since we do not know the boundary value of \( E_{1,3} \). To get around this, we use (37), which gives
\[ \partial_{x_3} E_{1,3} = \frac{\partial E_{1,3}}{\partial \nu} = \frac{1}{i\omega \varepsilon_0} \nabla \cdot \tilde{F}_2 \text{ on } \Gamma_1, \quad \partial_{x_3} E_{1,3} = -\frac{\partial E_{1,3}}{\partial \nu} = \frac{1}{i\omega \varepsilon_0} \nabla \cdot \tilde{F}_2 \text{ on } \Gamma_2. \]
This is a Neumann boundary condition for \( E_{1,3} \) on \( \partial \Omega \). Using the trace theorem again, we can transform the problem into one with zero Neumann boundary condition: Let \( \tilde{E} \in H^1(\mathbb{R}^3) \) with compact support be such that \( \partial \tilde{E}/\partial \nu = \frac{1}{i\omega \varepsilon_0} \nabla \cdot \tilde{F}_2 \text{ on } \partial \Omega \), and look for \( E_{1,3} = \tilde{E} + \hat{E} \), where \( \hat{E} \) now solves
\[ (-\Delta - k^2)\hat{E} = G_3 + (\Delta + k^2)\hat{E} \text{ in } \Omega, \quad \frac{\partial \hat{E}}{\partial \nu} = 0 \text{ on } \partial \Omega. \]
We further need a radiation condition for \( \hat{E} \): we can expand \( \hat{E} \) as a cosine series, and the coefficients of this series need to satisfy the same partial radiation conditions as stated above for the coefficients of \( E_{1,1}, E_{1,2} \). In order to find \( \hat{E} \), we employ the fundamental solution for zero Neumann boundary data,
\[ \Psi(x, y) = \sum_{m=1}^{\infty} -\frac{1}{2L} \cos \left( \frac{m\pi x_3}{L} \right) \cos \left( \frac{m\pi y_3}{L} \right) H^1_0(k_m|x'| - |y'|). \] (41)
We then get
\[ \hat{E}(x) = \int_{\Omega} \Psi(x, y)(G_3 + (\Delta + k^2)\hat{E}) \, dy, \]
and thus
\[ E_{1,3}(x) = \hat{E}(x) + \int_{\Omega} \Psi(x, y)(G_3 + (\Delta + k^2)\hat{E}) \, dy \in H^1(\Omega). \] (42)

In order to verify that \( E_1 = (E_{1,1}, E_{1,2}, E_{1,3}) \) found in this process is indeed a suitable solution for Maxwell’s equations, we compute \( \nabla \cdot E_1 \) to confirm that (37) is satisfied. Using the properties of the fundamental solutions \( \Phi \) and \( \Psi \), this is readily verified. Thus, \( E_1 \in H^1(\Omega)^3 \) and \( H_1 := \frac{1}{i\omega\mu_o}(\nabla \cdot E_1 - \hat{F}_1) \) are solutions to (33a)-(33d). Note that due to the symmetry of the system (33), we also have \( H_1 \in H^1(\Omega)^3 \).

**ME with nonconstant coefficients on \( \Omega_b \).** Now we consider the system (34) on the bounded \( C^{1,1} \) domain \( \Omega_b \). Note that since \( E_1|_{\partial \Omega_b} \in H^1(\Omega_b)^3 \), its tangential trace \( \nu \wedge E_1 \) belongs to \( TH^{1/2}(\partial \Omega_b) \). The system (34) is known to have a unique pair of solutions \( (E_2, H_2) \in H_{\text{curl}}(\Omega_b)^2 \) (this is true even if \( \Omega_b \) is a Lipschitz domain); due to the regularity of the domain and data in our situation, we can further improve the regularity of the solutions: Using (34b), and the identity (35) for \( \hat{F}_2 \), we compute the divergence of \( E_2 \) and find
\[ \nabla \cdot E_2 = \frac{1}{i\omega\gamma}[\nabla \cdot \hat{F}_2 - i\omega\gamma \cdot E_2] \in L^2(\Omega_b) \]
by our assumption on the regularity of \( \gamma \). So \( E_2 \in H_{\text{curl}}(\Omega_b) \cap H_{\text{div}}(\Omega_b) \) and \( \nu \wedge E_2 = \nu \wedge E_1 \in \text{TH}^{1/2}(\partial \Omega_b) \); this suffices to conclude that \( E_2 \in H^1(\Omega_b)^3 \) (see [GR86 Sec. I.3]). Similarly,
\[ \nabla \cdot H_2 = \frac{1}{i\omega\mu}[\nabla \cdot \hat{F}_1 - i\omega\mu \cdot H_2] \in L^2(\Omega_b), \]
\[ i\omega \nu \cdot (\mu H_1) = \nu \cdot (\nabla \wedge E_1 - \hat{F}_1) = -\nabla_{\partial \Omega_b} \cdot (\nu \wedge E_1) - \nu \cdot \hat{F}_1 = -\nabla_{\partial \Omega_b} \cdot (\nu \wedge E_2) - \nu \cdot \hat{F}_1 = i\omega \nu \cdot (\mu H_2), \]
so \( H_2 \in H_{\text{curl}}(\Omega_b) \cap H_{\text{div}}(\Omega_b) \), and \( \nu \cdot H_2 = \nu \cdot H_1 \in H^{1/2}(\partial \Omega_b) \), and we conclude \( H_2 \in H^1(\Omega_b) \).

**Unique solvability of (35).** In [LU10], a scalar equation analogous to (35) was treated by showing that the operator corresponding to \( K \) is compact and then using Fredholm theory to show unique solvability. I am in the process of analyzing the operator \( K : H^1(\Omega_b)^6 \to H^1(\Omega_b)^6 \) defined in (36) in order to apply an analogous argument in this situation. A first observation is that \( K \hat{F} \) satisfies a vector Helmholtz equation with right-hand side in \( L^2(\Omega)^6 \). Indeed, note that the identities
\begin{align*}
\nabla \cdot (E_2 - E_1) &= \nabla \cdot (H_2 - H_1) = 0, \quad (43) \\
\nabla \wedge (E_2 - E_1) &= i\omega \mu_o (H_2 - H_1), \quad (44) \\
\nabla \wedge (H_2 - H_1) &= -i\omega \varepsilon_o (E_2 - E_1) \quad (45)
\end{align*}
hold wherever \( \nabla \varphi \neq 0 \), since in this region we have constant parameters \( \mu = \mu_o \) and \( \gamma = \varepsilon_o \). Thus,
\[ \nabla \cdot (\nabla \varphi \wedge (E_2 - E_1)) = -\nabla \varphi \cdot (\nabla \wedge (E_2 - E_1)) = -\nabla \varphi \cdot i\omega \mu_o (H_2 - H_1), \]
and further
\[ -\nabla [\nabla \cdot (\nabla \varphi \wedge (E_2 - E_1))] = (\nabla \varphi \cdot \nabla) i\omega \mu_o (H_2 - H_1) + i\omega \mu_o ((H_2 - H_1) \cdot \nabla) \nabla \varphi + \nabla \varphi \wedge (\nabla \wedge i\omega \mu_o (H_2 - H_1)). \]
Using (45), the last term becomes \( k^2 \nabla \varphi \wedge (E_2 - E_1) \), so we obtain
\[ (-\Delta - k^2)(\nabla \varphi \wedge (E_2 - E_1)) = i\omega \mu_o \left( (\nabla \varphi \cdot \nabla) (H_2 - H_1) + ((H_2 - H_1) \cdot \nabla) \nabla \varphi \right), \]
and the right-hand side of this equation lies in \( L^2(\Omega_b)^3 \). Analogously one sees that \( \nabla \varphi \wedge (H_2 - H_1) \) satisfies a similar equation. If the boundary data can be shown to be in \( H^{3/2}(\partial \Omega)^6 \), elliptic regularity estimates would then yield that \( K \hat{F} \in H^2(\Omega_b)^6 \), which would in turn prove compactness of \( K \). At this point, however,
I haven’t been able to show that the boundary data satisfies the needed regularity condition. Right now I am working on this as well as testing other approaches for handling $K$.

Once well-posedness of the direct problem is established, in studying the inverse problem, the first step is to find an integral identity for each of the two settings described in [28] and [29]. Since data is known only on parts of the boundary, this formula will involve unknown boundary integrals, and a Carleman type estimate will be needed in order to control these boundary terms. The next step will be to construct the necessary CGO solutions, in the process of which I expect to employ reflection arguments similar to [LU10, KLU12]. Since reflecting the parameters along a boundary will yield extended parameters that are only Lipschitz continuous in general, results from Section 2 will be needed here.

References


