# THE REPRESENTATION RINGS OF $k[X]$

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Abstract. We give a short proof for the Clebsch - Gordan decompositions for the finite-dimensional modules over $k[X]$.

## 1. The representation ring of $k[X]$: the primitive case

Let $k$ be an algebraically closed field of characteristic zero. The structure of finitely generated $k[X]$ modules is well-known: a torsion-free module is free and an indecomposable torsion module is isomorphic to $J_k(\mu, m) := k[X]/(X-\mu)^m$ for some $\mu \in k$ and some natural number $m$. The modules $J_k(\mu, m)$ and $J_k(\mu', m')$ are not isomorphic if $(\mu, m) \neq (\mu', m')$. If the field $k$ is fixed we shall simply write $J(\mu, m)$. The isomorphism class of this module will be denoted $[J(\mu, m)]$ and the image of this module in any representation ring of $k[X]$ will be denoted $[\mu, m]$.

Viewed as a $k$-vector space, $J(\mu, m)$ has a standard basis $\{e_i := (X-\mu)^{i-1}\}_{i=1,\ldots,m}$. Since $(X-\mu)e_i = e_{i+1}$ for all $i \geq 1$ (assuming that $e_{m+1} = e_{m+2} = \ldots = 0$), we have that

$$Xe_i = \mu e_i + e_{i+1}$$

for all $i$. Hence, in this basis, $X$ acts on $J(\mu, m)$ as $\mu 1_m + D_m$, where $D_m$ is the nilpotent operator sending each $e_i$ to $e_{i+1}$.

Let $\mathcal{C}$ be the full subcategory of $k[X]$-mod consisting of modules which are finite-dimensional over $k$. It is immediate that $\mathcal{C}$ is closed under isomorphisms, finite direct sums, and the tensor product over $k$. Furthermore, by the structure theorem for finite torsion modules over a PID, $\mathcal{C}$ has the Krull - Remak - Schmidt property. Therefore, the representation ring $\mathcal{R}(\mathcal{C})$ is a free $\mathbb{Z}$-module on the elements $[\mu, m]$. Our goal in this section is to describe the multiplicative structure of the representation ring $\mathcal{R}(\mathcal{C})$ of $k[X]$ corresponding to the primitive product $A \otimes 1 + 1 \otimes B$. In the next section we shall solve the same problem for the Kronecker product.

Given $k[X]$-modules $M := J(\mu, m)$ with standard basis $e_i, i = 1, \ldots, m$, and $N := J(n, \nu)$ with standard basis $f_j, j = 1, \ldots, n$, we define an action of $X$ on $M \otimes_k N$ by the matrix $A \otimes 1 + 1 \otimes B$, where $A$ and $B$ are the matrices corresponding to $M$ and $N$. In the basis $e_{i,j} := e_i \otimes f_j$, it is given by the operator

$$(\mu 1_m + D_m) \otimes 1_n + 1_m \otimes (\nu 1_n + D_n) = (\mu + \nu) 1_{mn} + D_m \otimes 1_n + 1_m \otimes D_n.$$

Here $(\mu + \nu)1_{mn}$ is the *semi-simple* part of the operator $X : M \otimes N \to M \otimes N$ and

$$D := D' + D'' : e_{i,j} \mapsto e_{i+1,j} + e_{i,j+1}, \tag{1.1}$$

where $D' := D_m \otimes 1_n$ and $D'' := 1_m \otimes D_n$, is the *nilpotent* part of $X$. In short,

$$Xe_{i,j} = (\mu + \nu)e_{i,j} + e_{i+1,j} + e_{i,j+1}. \tag{1.2}$$

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**Theorem 1.** *For $\mu, \nu \in k$ and positive integers $m \le n$, the multiplication in $\mathcal{R}(\mathcal{C})$ is given by the formula*

$$[\mu, m] \cdot [\nu, n] = \sum_{i=0}^{m-1} [\mu + \nu, n + m - 1 - 2i].$$

*Proof.* We proceed by induction on $m$, keeping $\mu, \nu$, and $n$ fixed. For $m = 1$ the theorem is trivial. Suppose that $m = 2$. Using the basis $(e_{i,j})$ defined above, we introduce the vector subspaces

$$M_{n+1} := \langle e_{1,j} + (j-1)e_{2,j-1} \rangle_{j=1,\ldots,n+1}$$

and

$$M_{n-1} := \langle e_{1,j} - (n-j+1)e_{2,j-1} \rangle_{j=2,\ldots,n}$$

of $J(\mu, 2) \otimes J(\nu, n)$. Using formula (1.2), one checks that $M_{n+1}$ and $M_{n-1}$ are in fact $k[X]$-submodules isomorphic to $J(\mu+\nu, n+1)$ and, respectively, $J(\mu+\nu, n-1)$. It remains to show that they have a trivial intersection. As vector subspaces, both submodules are graded by the index $j$. Therefore, it suffices to show that their intersection is trivial in each degree. This is immediate since

$$\begin{vmatrix} 1 & 1 \\ j-1 & -(n-j+1) \end{vmatrix} = -n \neq 0.$$

Now assume that the theorem is true for all values of $m \le l - 1$ and suppose that $l \le n$. By the associativity of multiplication,

$$([\nu, n] \cdot [\mu, l-1]) \cdot [0, 2] = [\nu, n] \cdot ([\mu, l-1] \cdot [0, 2]).$$

By the induction assumption, the left-hand side equals

$$\sum_{i=0}^{l-2} ([\mu + \nu, n + l - 1 - 2i] + [\mu + \nu, n + l - 3 - 2i]),$$

whereas the right-hand equals

$$[\nu, n] \cdot [\mu, l] + \sum_{i=0}^{l-3} [\mu + \nu, n + l - 3 - 2i].$$

The desired result now follows. □

## 2. The representation ring of $k[X]$: the Kronecker product

In this section we shall describe the multiplicative structure of the representation ring $\mathcal{R}(\mathcal{C})$ of $k[X]$ corresponding to the Kronecker product. We continue to assume that $k$ is an algebraically closed field. In the notation of the previous section, given $k[X]$-modules $M := J(\mu, m)$ with standard basis $e_i, i = 1, \ldots, m$, and $N := J(n, \nu)$ with standard basis $f_j, j = 1, \ldots, n$, we define an action of $X$ on $M \otimes_k N$ by the matrix of $A \otimes B$, where $A$ and $B$ correspond to $M$ and $N$. In the basis $e_{i,j} := e_i \otimes f_j$, is given by the operator

$$(\mu 1_m + D_m) \otimes (\nu 1_n + D_n) = (\mu\nu) 1_{mn} + \mu 1_m \otimes D_n + \nu D_m \otimes 1_n + D_m \otimes D_n.$$

Here $(\mu\nu)1_{mn}$ is the *semi-simple* part of the operator $X : M \otimes N \to M \otimes N$ and

$$D := D' + D'' + D''' : e_{i,j} \mapsto \nu e_{i+1,j} + \mu e_{i,j+1} + e_{i+1,j+1},$$

where $D' := D_m \otimes 1_n$, $D'' := 1_m \otimes D_n$, and $D''' := D_m \otimes D_n$ is the *nilpotent* part of $X$. In short,

$$Xe_{i,j} = (\mu\nu)e_{i,j} + \nu e_{i+1,j} + \mu e_{i,j+1} + e_{i+1,j+1}. \tag{2.1}$$

**Theorem 2.** *For $\mu, \nu \in k$ and positive integers $m \leq n$, the multiplication in $\mathcal{R}(\mathcal{C})$ is given by the formulas:*

(1) $[\mu, m] \cdot [\nu, n] = \sum_{i=0}^{m-1} [\mu\nu, n+m-1-2i]$ *if $\mu \neq 0$ and $\nu \neq 0$,*
(2) $[\mu, m] \cdot [0, n] = m[0, n]$ *if $\mu \neq 0$,*
(3) $[0, m] \cdot [\nu, n] = n[0, m]$ *if $\nu \neq 0$,*
(4) $[0, m] \cdot [0, n] = (n - m + 1)[0, m] + 2 \sum_{i=1}^{m-1} [0, i]$.

*Proof.* We begin with case 1, when both eigenvalues are different from zero. Our argument will again use induction on $m$, similar to the primitive case. If $m = 1$, then $Xe_{1,j} = (\mu\nu)e_{1,j} + \mu e_{1,j+1}$ and the vectors $e_{1,1}, \mu e_{1,2}, \ldots, \mu^{n-1} e_{1,n}$ form a Jordan basis for $[\mu, 1] \cdot [\nu, n]$. If $m = 2$, we introduce two vector subspaces $M_{n+1}$ and $M_{n-1}$ of $J(\mu, 2) \otimes J(\nu, n)$, defined as follows. The subspace $M_{n+1}$ is spanned by the $n+1$ vectors $e_{1,1}$ and $\mu^{i-1}(i\nu e_{2,i} + ie_{2,i+1} + \mu e_{1,i+1}), i = 1, \ldots, n$. Since both $\mu$ and $\nu$ are different from zero, this is a Jordan basis of length $n+1$ with eigenvalue $\mu\nu$. Therefore $M_{n+1}$ is a $k[X]$-submodule of $J(\mu, 2) \otimes J(\nu, n)$ isomorphic to $J(\mu\nu, n+1)$. Notice that the socle of $M_{n+1}$ is spanned by $e_{2,n}$. To define the other subspace, $M_{n-1}$, we choose two scalars $\alpha, \beta \in k$ and take the linear span of the vectors

$$g_i := (\alpha\mu^{i-1} + (i-1)\beta\nu\mu^{i-2})e_{2,i} + (i-1)\beta\mu^{i-2}e_{2,i+1} + \beta\mu^{i-1}e_{1,i+1},$$

where $i = 1, \ldots, n-1$. We claim that a suitable choice of $\alpha$ and $\beta$ will make this system of vectors into a Jordan basis of length $n-1$ with eigenvalue $\mu\nu$. Indeed, $Dg_i = g_{i+1}$ for each $i \geq 1$. Thus we only have to check that $g_{n-1} \neq 0$ and $Dg_{n-1} = 0$ for a suitable choice of $\alpha$ and $\beta$. Since $\mu \neq 0$, to satisfy the first condition it suffices to require that $\beta \neq 0$, as this would make the coefficient of $e_{1,n}$ in $g_{n-1}$ different from zero. As $Dg_{n-1}(\alpha\mu^{n-1} + (n-1)\beta\nu\mu^{n-2})e_2, n$, the second condition can also be satisfied by a suitable choice of $\alpha$ since $\mu \neq 0$. Thus we can assume that $M_{n-1}$ is a $k[X]$-submodule of $J(\mu, 2) \otimes J(\nu, n)$ isomorphic to $J(\mu\nu, n-1)$. Notice that the simple socle of $M_{n-1}$ is spanned by $g_{n-1}$, whose coefficient in $e_{1,n}$ is different from zero. Therefore the socles of $M_{n+1}$ and $M_{n-1}$ do not intersect. Thus the two submodules do not intersect, and we have that $J(\mu, 2) \otimes J(\nu, n)$ is isomorphic to $J(\mu\nu, n+1) \amalg J(\mu\nu, n-1)$. The rest of the induction proof is identical to the one given in the previous section. We now consider case 2: $\mu \neq 0$ and $\nu = 0$. In this case, for each $i = 1, \ldots, m$, we consider the vector subspace $M_i$ of $J(\mu, m) \otimes J(0, n)$ spanned by the vectors $e_{i,1}, De_{i,1}, D^2 e_{i,1}, \ldots, D_i^{n-1} e_{i,1}$. Again this is a $k[X]$-module. Its socle is a linear combination of the vectors $e_{i,n}, \ldots, e_{m,n}$ with the coefficient in $e_{i,n}$ being $\mu^{n-1} \neq 0$. This observation has two consequences. First is that the above vectors form a Jordan basis of length $n$ and, therefore, $M_i$ is a nilpotent Jordan block of dimension $n$ for each $i = 1, \ldots, m$. Secondly, the socles of the $M_i$'s are pairwise distinct and, therefore, $\sum_{i=1}^{m} M_i$ is a direct sum. Comparing the dimensions we have that $[\mu, m] \cdot [0, n] = m[0, n]$. In case 3, the proof is identical to the one just given. We now consider case 4: both $\mu$ and $\nu$ are zero. Under this assumption, $De_{i,j} = e_{i+1,j+1}$ for each pair of indices $i$ and $j$. Direct examination now shows that: the vector subspaces $M_j :=< e_{1,j} De_{1,j}, \ldots, D^m e_{1,}j >$ $, j = 1, \ldots, n-m+1$ are Jordan blocks of length $m$, the vector subspaces $N_i :=<$ $e_i, 1, De_{i,1}, \ldots, D^{m-i} e_{i,1} >, i = 2, \ldots, m$ are Jordan blocks of length $m-i+1$, and the vector subspaces $P_j =< e_{1,n-m+j}, De_{1,n-m+j}, \ldots, D^{m-j} e_{1,n-m+j} >, j =$ $2, \ldots, m$ are Jordan blocks of length $m-j+1$. The desired result now follows. □

**Remark 1.** *Without the assumption $m \leq n$, there is no distinction between case 2 and case 3.*

## References

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