# Maximin Designs for Testing Degree of a Polynomial

Mikhail B. Malyutov
Mathematics Dept., Northeastern University, Boston, MA 02115, USA

$e - mail: \ mltv@neu.edu$

**Abstract**

Let an algebraic or trigonometric polynomial $\eta_d(x)$ of degree $d$ be measured upon a closed region $X$ with homoscedastic errors. We test the hypothesis $H : \theta_{(1)} = 0$, where $\theta_{(1)}$ is the vector of monomial coefficients of degree exactly $d$. Let $\Delta(\epsilon, \ \theta_{(1)})$ be the non-centrality parameter of the $F$-test for $H$ depending on the approximate experimental design $\epsilon$ and the vector $\theta_{(1)}$. The aim of this paper is to construct designs maximizing $inf_{||\theta_{(1)}||=1}\Delta(\epsilon, \ \theta_{(1)})$ for a series of symmetric sets $X$ from $\mathbf{R^k}$. Here $||\theta_{(1)}||$ is mostly the Euclidean norm of the vector $\theta_{(1)}$.

**Keywords**: testing significance of high-order coefficients, multinomial regression, non-centrality parameter, power of the test.

## 1 Introduction.

Approximating an unknown response function by a high-order polynomial in the whole operability region with the aim of searching its optimum is considered inappropriate after the pioneering Box and Wilson's paper [1] due to an excessive amount of experiments required for getting usually highly correlated and poorly interpretable parameter estimates.

An active experimental strategy is usually applied consisting of several stages.

During the first stage the response surface is locally fitted as multi-linear one yielding an estimate of its gradient to start a steepest ascent along the local response until a response's stationary region is reached.

Then the second stage of experimentation is initiated to fit a second order model for the response inside the stationary region for estimating the sensitivity of the response to small parameter variation. Inadequacy of the linear approximation signals about the necessity to the above change of the fitting strategy. For testing adequacy [1] recommended the complementing symmetric designs used for fitting first order model with several central points to enable a reliable estimate for the sum of parameters describing quadratic terms of the response approximation.

This simple convenient and intuitive recommendation has its drawback: when the above sum of parameters vanishes or is small, the experiments fail to detect inadequacy even if some of the quadratic parameters are large.

Thus more reliable designs are appropriate for this aim having a guaranteed power of detecting whatever quadratic deviation from the multi-linear model, if a saddle-type local behavior of the response cannot be excluded beforehand.

Constructing designs maximizing minimal power of the F-test for adequacy of multi-linear model on a cube or a ball is described in section 2. We study designs for a general discrimination problem between degrees of univariate polynomials in section 3 which may be considered as tutorial preparing for testing insignificance of all highest order monomials of multinomial regression in the following sections, although our approach based on Chebyshev polynomials might be new. Similar designs maximizing the minimal power of the F-test for discriminating between multiharmonic models is dealt with in the final section.

Now we formulate our problem more accurately.

Let the vector of measurements $y = (y_1, y_2, \ldots, y_N)^T$ be distributed $N(\eta(\theta),\ \sigma^2 I)$, where $\eta(\theta) = (\eta(\mathbf{x}(1), \theta), \ldots, \eta(\mathbf{x}(N), \theta))^T$, $(\eta(\mathbf{x}, \theta) = \theta^T f(\mathbf{x})$, $\theta$ is a vector of unknown parameters, $\mathbf{x}(i)$ is the value of controlled variables in the $i^{th}$ experiment, $\mathbf{x}(i) \in X$, $I_N$- is an $N \times N$ identity matrix. Let it be known that $\eta$ is well approximated by an algebraic or trigonometric polynomial of degree $d$, but we anticipate that a multinomial of degree $d-k$ will suffice. It seems reasonable at first to place measurements so as to test the hypothesis about degree of response to estimate optimally the multinomial coefficients afterwards.
We test the hypothesis $H$ by the $F$-test in view of its optimal properties [5]. The power of the $F$-test is an increasing function of the non-centrality parameter $\Delta$ which depends on the design $\epsilon$ and the vector $\theta_{(1)}$ of monomial coefficients of degrees from $d-k+1$ to $d$:

$$\sigma^2 \Delta(\epsilon,\ \theta_{(1)}) = \theta_{(1)}^T\ D_1^{-1}(\epsilon)\ \theta_{(1)} \tag{1}$$

where $D_1(\epsilon)$ denotes the central square block of a covariance matrix of the least square estimates (LSE) for $\theta$ related to the parameters $\theta_{(1)}$ (see [6],[11].
The matrix $D_1$ is inversely proportional to $N$ with a matrix coefficient depending on frequencies of experiments at points $x_i$. Therefore to get rid of the unnecessary parameter $N$ during optimization and make the design set closed and convex we consider an *arbitrary probability measure on $X$ as an approximate experimental design* and by $D_1$ we mean the square block of LSE's normalized covariance matrix corresponding to $\theta_{(1)}$ (discussion of these notions see e.g. in [6],[11]).
Choosing a certain norm $||\theta_{(1)}||$ of the vector $\theta_{(1)}$ we call design maximizing $\inf_{\theta} \Delta(\epsilon,\ \theta_{(1)})$ *maximin* design. Here $\Theta = \{\theta_{(1)} :\ ||\theta_{(1)}|| = 1\}$, note that $\Theta$ is *not* a convex set.

We find maximin designs for the following cases:

(1) $X$ is an n-dimensional cube: $|x_i| \leq 1$, $i = 1, 2, \ldots, n$, $\eta(x,\ \theta)$ is a multinomial of degree $d$, $k = 1$ or $2$, $||\theta_{(1)}||$ is the Euclidean norm of $\theta_{(1)}$.

(2) $X$ is an n-dimensional ball: $\sum_{i=1}^{n} x_i^2 \leq 1,\ \eta(x,\ \theta)$ is a multinomial of degree $d,\ k = 1$ or $2, anormof ||\theta_{(1)}||$ is invariant under rotations.

(3) $X$ is an n-dimensional torus: $|t_i| \leq \pi$, the points $(t_1, t_2, \dots, t_{m-1}, \pm\pi, t_{m+1}, \dots, t_n)$ are considered as identical for all $m, x_1, \dots, x_{m-1}, x_{m+1}, x_n :$

$$\eta(x,\ \theta) = \sum_{0<|\nu|\leq d} [a_\nu cos\ (\nu^T t) + b_\nu sin\ (\nu^T t)] + \frac{a_0}{\sqrt{2}} \tag{2}$$

where $\nu = (\nu_1, \dots, \nu_n)^T,\quad 0 \leq \nu_i \leq d,\ |\nu| = \sum \nu_i,\ (t_1, \dots, t_n)^T,\ ||\theta_{(1)}||$ is the sum of squares of the components of $\theta_{(1)}$ and the vector $\theta_{(1)}$ includes simultaneously $a_\nu,\ b_\nu$ for a certain set $A$ of the indices $\nu,\ 0 < \nu \leq d$.

The results have been announced in [7] which contained many typos (due to the fact that the proofs of this note were not taken into account), corrected in [8]. More detailed proofs were published in hardly accessible collections in Russian [9], [10]. In [9] and in [7] our designs were called (in a bit misleading way) minimax ones.

# 2 Direct Evaluation of Maximin Designs, $d = 2, k = 1$

We first study a particular case of multinomial regression on the unit cube and ball in $\mathbf{R}^n, n \in \mathbf{N}$, for $d = 2,\ k = 1$ using direct matrix calculations. The response function is assumed to be

$$\eta(\mathbf{x}) = \theta_0 + \sum_{1\leq i\leq n} \theta_i x_i + \sum_{1\leq i\leq j\leq n} \theta_{ij} x_i x_j := \theta_{(0)}\mathbf{x}_0 + \theta_{(1)}\mathbf{x}_{(1)}, \mathbf{x} := (x_1, \dots, x_n) \in X.$$

Here:

$$\theta_{(0)} := (\theta_0, \dots, \theta_n), \theta_{(1)} := (\theta_{11}, \theta_{12}, \dots, \theta_{nn}),$$

$$\mathbf{x}_{(0)}^T := (1, \dots, x_n), \mathbf{x}_{(1)} := (x_1^2, x_1x_2 \dots, x_n^2), T$$

is a transposition sign and powers are enumerated according to the lexicographic ordering.

Given a sequence of design points $\mathbf{x}(u) \in X, u = 1, \dots, N$ and corresponding measurements $y_u$, the tests $\chi^2$ and $F$ are based on comparison of the residual sum of squares

$$S_R := \sum_{u=1}^{N} p_u(\hat{\eta}(\mathbf{x}(u)) - y_u)^2$$

with respectively the variance of measurement errors $\sigma^2$, or its independent (of the design space) estimate, see e.g. [12]. Here $p_u$ is the frequency of approximate design $\epsilon$ points at $\mathbf{x}_u, u = 1, \dots, N$. The powers of both tests are increasing functions of

$$\Delta(\epsilon, \theta_{(1)}) := \sigma^{-2} \sum_{u=1}^{N} (\mathbf{E}\hat{\eta}(\mathbf{x}(u)) - \eta(\mathbf{x_u}))^2 p_u.$$

Introduce the partition of the *design matrix* $X$:

$$X_0 := ||p_u^{1/2} \mathbf{x}_{(0)}(u), u = 1, \dots, N||,$$
$$X_1 := ||p_u^{1/2} \mathbf{x}_{(1)}(u), u = 1, \dots, N||.$$

It is well-known ([12]) that

$$\mathbf{E}\theta_{(0)} = \theta_{(0)} + A\theta_{(1)},$$

where $A := (X_0^T X_0)^{-1} X_0^T X_1$. As a corollary, the noncentrality parameter $\Delta$ admits an alternative expression:

$$\Delta(\epsilon, \theta_{(1)}) = \sigma^{-2} \theta_{(1)}^T B \theta_{(1)}, \tag{3}$$

where $B := (A^T X_0 - X_1)(X_0^T A - X_1^T) = D_1^{-1}$ by the Frobenius formula:

$$D_1 = X_1^T X_1 - (X_0 X_1)^T (X_0^T X_0)^{-1} (X_0 X_1) \tag{4}$$

for the inverse to the block $D_1$ of the variance-covariance matrix $D$ (which in its turn is the inverse to the *information matrix* $M := X^T X$), see e.g. [2]. (3) is quadratic in $\theta_{(1)}$, shows independence of $\Delta$ from $\theta_{(0)}$ and has the following Approximation Theory interpretation which will prove useful very soon:

**Corollary**. If $\bar{\eta}(\mathbf{x})$ is the least squares interpolating polynomial (of degree 1) for $\eta(\cdot)$ over points $\mathbf{x}(u)$ with weights $p_u, u = 1, \dots, N$, then

$$i.\bar{\eta}(\mathbf{x}(u)) - \eta(\mathbf{x}(u))) := \delta(x, \epsilon, \theta_{(1)})$$

is a linear form in $\theta_{(1)}$, and the interpolation error is (see [3])

$$\sum_{u=1}^{N} p_u [\bar{\eta}(\mathbf{x}(u)) - \eta(\mathbf{x}(u)))]^2 = \sigma^2 \Delta(\epsilon, \theta_{(1)}).$$

Following classical J. Kiefer's argument, we prove

**Lemma 2.1.**. *There exists a symmetric maximin design, i.e. invariant with respect to all permutations of axes and change of their signs.*

**Proof**. If $\epsilon$ is a maximin design, and $\bar{\epsilon}$ is the design obtained by its averaging over all permutations and sign changes, then $B_1(\bar{\epsilon}) - B_1(\epsilon)$ is nonnegative-definite ([6],[11]), and therefore

$$\Delta(\bar{\epsilon}) - \Delta(\epsilon) \geq 0$$

as difference between minimal eigenvalues of $B_1(\bar{\epsilon})$ and $B_1(\epsilon)$. It remains to note that the design $\bar{\epsilon}$ is symmetric.

It is obvious that the information matrix of a symmetric design has only the following distinct entries:

$$a = \sum x_i^2(u)p_u, b = \sum x_i^4(u)p_u, c = \sum x_i^2(u)x_j^2(u)p_u, i < j.$$

All other moments vanish for symmetric designs (which we now denote simply $\epsilon$).

After elementary calculations we come to the following block structure of $B(\epsilon)$: there is a central $n$-block

$$J = (c - a^2)\mathbf{i}_n\mathbf{i}_n{}^T + (b - c)I_n,$$

corresponding to $\theta_{ii}, i = 1, \ldots, n$, all other off-diagonal entries are 0, while other diagonal entries are $c$. Here $\mathbf{i}_n$ is the $n$-vector with all entries 1, and $I_n$ is the identity matrix of order $n$.

It is easy to see that $\mathbf{i_n}$ is an eigenvector of $J$ (and consequently of $B$) with the eigenvalue

$$\lambda_1 = b + (n - 1)c - na^2.$$

The rank of $J - (b - c)I_n$ equals one implying that

$$\lambda_2 = b - c$$

is the eigenvalue of $J$ (and B) of multiplicity $n - 1$.

Finally,

$$\lambda_3 = c$$

is obviously an eigenvalue of $B$ of multiplicity $n(n - 1)/2$. Thus we have proved

**Lemma 2.2.** For every symmetric design $\epsilon$ its noncentrality parameter equals

$$\sigma^{-2} \min\{c, b - c, b + (n - 1)c - na^2\}.$$

## 2.1 Symmetric Maximin Design for the Unit Ball

For designs supported by the unit ball $\sum x_i^2 \leq 1$ we have an obvious inequality

$$a \geq \sum p_u x_i^2(u)(x_i^2(u) + \sum_{j \neq i} x_j^2(u)) = b + (n - 1)c.$$

Hence $\Delta(\epsilon) \leq \lambda_1 \leq a - na^2$. Besides, Lemma 2.2 implies that $\Delta(\epsilon) \leq c, \Delta(\epsilon) \leq b - c$.

Thus $b \geq 2\Delta(\epsilon), a \geq b + (n - 1)c \geq (n + 1)\Delta(\epsilon)$ implying $a \geq 2n^{-1}$ and $\Delta(\epsilon) \leq (n + 1)(\Delta(\epsilon) - n(+1)\Delta^2(\epsilon))$.

Combining all preceding bounds, we conclude that for every symmetric design in the unit ball it holds

$$\Delta(\epsilon) \leq (n + 1)^{-2}.$$

Now we look for a design attaining equality in the last inequality (which means that the design constructed is maximin) to consist of:

i. the weight of each inscribed cube vertex ($x_i = n^{-1/2}$) is $s2^{-n}$;

ii. the weight of each star point ($|x_i| = 1, x_j = 0, j \neq i$) is $t/2n$;

iii. the weight of the center is $r, s + t + r = 1$.

This design $\epsilon^*$ has parameters

$$a = (s + t)/n, b = sn^{-2} + t/n, c = sn^{-2}.$$

Choosing

$$s = (n/(n + 1))^2, t = n(n + 1)^{-2}, r = (n + 1)^{-1},$$

we make all the eigenvalues of $B$ equal $(n + 1)^{-2}$, and the noncentrality parameter

$$\Delta(\epsilon^*, \theta_{(1)}) = ||\theta_{(1)}||^2\sigma^{-2}(n + 1)^{-2}.$$

Since the non-approximate version of $\epsilon^*$ contains excessively many points (already 72 for $n = 3$), it makes sense to approximate $\epsilon^*$ by a more economical design. For example the design of 8 measurements in the center, single measurement in every star point and 2 measurements at each vertex of the inscribed cube well approximates the best value of the noncentrality parameter.

## 2.2 Symmetric Maximin Designs in the Unit Cube

On the unit cube $\{\mathbf{x} : \max |x_i| \leq 1\}$ the moments of a symmetric design $\epsilon$ satisfy obvious inequality $a \geq b$. Thus

$$\Delta(\epsilon) \leq \lambda_1 \leq b + (n - 1)c - nb^2.$$

Putting $b - 1/2 + \beta, c = 1/4 + \gamma$, we get

$$\Delta(\epsilon) \leq \lambda_1 \leq \min\{\gamma, \beta - \gamma, (\gamma - \beta)(n - 1) - n\beta^2\} \leq 1/4.$$

Below we construct designs $\epsilon^*$ with all $B$'s eigenvalues 1/4, i.e. maximin ones, satisfying $a = b = 1/2, c = 1/4$. Hence the noncentrality parameter $\Delta(\epsilon^*) = ||\theta_{(1)}||/4\sigma^2$.

Consider the family of designs with the center weight $p_n$, equal weights of $N_r = 2^{n-r}\binom{n}{r}$ centers of $r$-dimensional faces are $p_r, r = 0, 1, \ldots, n$, the weight of each cube vertex is $p_0$, such that $\sum_{r=0}^{n} p_r N_r = 1$. It is obvious that $a = b$ for such designs. Let us evaluate moments of first two coordinates for definiteness. Since only those terms enter $a$ (respectively $c$) which have a nonzero first coordinate (respectively two first coordinates), we get

$$a = b = \sum_{r=0}^{n-1} p_r 2^{n-r}\binom{(n-1)}{r},$$

$$c = \sum_{r=0}^{n-2} p_r 2^{n-r} \binom{(n-2)}{r}$$

Denoting $q_r := 2^{n-r} p_r$, it is sufficient to solve system of equations:

$$\sum_{r=0}^{n} q_r \binom{n}{r} = 1,$$

$$\sum_{r=0}^{n-1} q_r \binom{(n-1)}{r} = 1/2,$$

$$\sum_{r=0}^{n-2} q_r \binom{(n-2)}{r} = 1/4,$$

which is satisfied if either:

i. we choose

$$q_r = 2^{-n}, r = 0, 1, \ldots, n.$$

This follows from the well-known binomial identities

$$\sum_{r=o}^{m} \binom{m}{r} = 2^m,$$

ii. or we choose $q_r = 2^{-(n-1)}$ for odd r, and 0 otherwise, its maximin property is proved similarly to above; or

iii. interchanging odd and even dimensions in ii.

The options ii. and iii. are more economical requiring less different experiments. For $n = 3$ the even series iii. requires 4 central measurements and measurements in each of 1-faces (edges), altogether 16 measurements. For n=5 the following maximin design belongs to neither of preceding series: 16 central measurements and one at the center of every 2-faces, altogether 96 measurements.

## 2.3 Some Additional Facts about Maximin Designs

Due to a simple structure of their Covariance matrices, the above symmetric maximin designs admit very simple formulas for MLE. Say,

$$\hat{\theta}_i = a^{-1} \sum y_u x_i(u) p_u, \hat{\theta}_{ij} = Var(\hat{\theta}_{ij}) \sum y_u x_{ij}(u) p_u, i < j,$$

$$\hat{\theta}_{ii} \sum p_u y_u [4x_{ii}^2(u) - 2], \hat{\theta}_{ii} = \sum p_u y_u (n+1)[(n+1)x_{ii}^2 - 1]$$

for cube and ball respectively, etc.

Maximin designs need not be symmetric: there is a maximin design in the unit circle supported by the vertices of the regular inscribed $m$-polyhedron with $m > 4$, the center and the vertices of the inscribed square. Function $\min_\Theta \Delta(\cdot, \theta_{(1)})$ need not be convex. However, the following assertion holds

**Proposition**. *All designs $\epsilon$ which are maximin and first order orthogonal (i.e. all first moments vanish and second order moments are constant (= a)), have the same information matrix .*

**Proof**. Using Frobenius formula and expression $X_0^T X_0 = diag(1, a, \ldots, a)$, we get the following formula:

$$\sigma^2 \Delta(\epsilon, \theta_{(1)}) = -a^2 (\sum \theta_{ii}) - a^{-1} \sum_{s=1}^{n} (\sum_{j \geq i} [ijs]_\epsilon)^2 + \sum_{j \geq i} \sum_{t \geq s} \theta_{ij} \theta_{st} [ijst]_\epsilon, \tag{5}$$

where $[\cdot]_\epsilon$ mean corresponding moments of design $\epsilon$. First note that introducing for every $i < j$ vector $\theta_{ij}$ of coefficients of $\eta(\mathbf{x}) = x_i x_j$ we have

$$\Delta(\epsilon, \theta_{(1)}) \leq \Delta(\epsilon, \theta_{ij}) = [iijj] - a^{-1} \sum_{s=1}^{n} [ijs]^2.$$

The symmetrized design $\bar{\epsilon}$ is also maximin and has the same second order moment. Also $[iijj]_{\bar{\epsilon}}$ is the average of $[iijj]_\epsilon$. Hence coincidence of their non-centrality parameters implies $[iijj]_\epsilon = c, [ijs] = 0$.

Now, comparing similarly noncentrality parameters of $\eta(\mathbf{x}) = n^{-1/2} \sum x{-}i^2$ for $\epsilon$ and its average, we conclude that $[iii]_\epsilon = 0, [iiii]_\epsilon = b$. Hence all third order moments vanish implying that all moments $[ijst]_\epsilon$ with at least each coordinate entering odd number of times vanish.

Let us study now properties of a much more economical Box's design $\epsilon_p$ consisting of the complete factorial experiment (CFE) with weight p and $q = 1 - p$ of experiments in the center. It is symmetric, and $a = b = c = p$. Thus $\lambda_1 = n(p - p^2) \leq n/4, \lambda_2 = 0, \lambda_3 = p$, and the optimal value for $p$ is 1/2 for $n > 1$. It is easy to see that hyperbolic responses, say $x_1^2 - x_2^2$, cannot be distinguished from constants using this design. However, if experimenters have good reasons to believe that the response in study looks like a hill, then Box's design is quite good in testing discrepancy from the first order model which is demonstrated by the following bound for the unit cube:

*In the class $\Theta_+ = \{\theta_{ii}, i = 1, \ldots, k,$ have the same sign $\}$, it holds:*

$$\sigma^2 \Delta(\epsilon_{1/2}, \theta_{(1)}) = k(\sum \theta_{ii})^2/4 + \sum_{i<j} \theta_{ij}^2/2 \geq \sum_{i \leq j} \theta_{ij}^2 = ||\theta_{(1)}||/4 = 1/4.$$

$\epsilon_{1/2}$ outperforms the maximin design even stronger for $\theta_{(1)} \in \Theta_+$ on the unit ball. Apparently, similar results hold, if the CFE is replaced with its suitable orthogonal fraction.

Finally, an intuitively appealing sequential strategy is proposed: after an orthogonal design $\epsilon_0$ fails to provide a sufficiently steep ascent locally it must be complemented with the same total amount of the design points in the center of $\epsilon_0$. It is fairly likely that already this design will detect a significant deviation from the first order model. If not, it is safe to add points constituting together with the old ones a design with guaranteed detection properties such as the maximin design, unless it is strongly believed that no saddles can happen.

## 2.4 Auxiliary Results

In the next sections we use the equivalence theorem [2] for a number of optimality criteria. Let us give the corresponding formulations:

**Theorem 2.1.**

(i) $\phi(M(\epsilon^\star)) = \max \phi(M(\epsilon))$ *for*

(1) $\phi_1(M) = D_{pp} = D\hat{\theta}_p$ *(the best estimate for the parameter $\theta_p$)*

*(2)* $\phi_2(M) = |D_1|$ *where $D_1$ is a submatrix of $D$ corresponding to a subvector $\theta_{(1)}$ of parameters (truncated D-optimality)*

*(3)* $\phi_3(M) = \theta_{(1)}^T D_1^{-1} \theta_{(1)} = \Delta(\epsilon,\ \theta_{(1)})$ *(maximization of the non-centrality parameter, the subvector of the parameters $\theta_{(1)}$ being given)*

*(4)* $\phi_4(M) = \int\ \Delta(\epsilon,\ \theta_{(1)})\ d\mu(\theta_{(1)})$ *(maximization of the weighted non-centrality parameter), $d\mu$ is the uniform measure on* $\Theta = \{\theta_{(1)} :\ ||\theta_{(1)}|| = 1\}$

*if respectively*:

(1) $\max_{x\in X}\ (\sum_{\alpha=1}^{p}\ D_{\alpha p}(\epsilon^\star) f_\alpha(x))^2 = D_{pp}(\epsilon^\star)$,

(2) $\max_{x\in X}\ f^T(x)\tilde{D}(\epsilon^\star)f(x) = dim\ \theta_{(1)}$, *in matrix $\tilde{D}$ block $D_1$ is replaced with* $D_1 - [X_1^T X_1]^{-1}$.

*(3)* $\max_{x\in X}\ \delta^2(x, \epsilon^\star, \theta_{(1)}) = \Delta(\epsilon^*, \theta_{(1)})$,
$\delta^2(x, \epsilon^*, \theta_{(1)}) = \min_{\beta_{(1)}=0} \int (\eta(x,\theta) - \eta(x,\beta))^2 d\epsilon = \int (\eta(x,\ \theta) - \eta_0(x,\epsilon,\theta))^2 d\epsilon$, *where on the function $\eta_0(x,\epsilon,\theta)$ the minimum is attained in the previous equality;*

*(4)* $\max_{x\in X}\ \int_\Theta \delta^2(x, \epsilon^\star, \theta_{(1)}) d\mu(\theta_{(1)}) = \int_\Theta \Delta(\epsilon^\star,\ \theta_{(1)}) d\mu(\theta_{(1)})$.

*(ii) supp $\epsilon^\star$ of the optimal design is contained in the set where maximum on $X$ is attained in the previous equalities.*

All cases of the theorem follow from general results in [11]. The proof of point (3) is described in [3]. Here we shall outline the proof of statement (4). The convexity of $\phi_4$ as functional of $M(\epsilon)$ follows from the convexity of the matrix $D_1^{-1}$ (e.g. see [3]). Besides $\phi_4(M)$ and the norm of its gradient increase to $\infty$ as $M \to M_-$, where $M_-$ is the complement to the set $M_+$ of designs for which $\theta_{(1)}$ is estimable.

The function $\phi_4$ is differentiable for $M \in M_+$ and it holds

$$\frac{\partial\phi(\epsilon_\alpha)}{\partial\alpha}|_{\alpha=0} = \int (\Delta(\epsilon_0,\ \theta_{(1)}) - \int \delta^2(x,\epsilon,\theta) d\epsilon_1) d\mu(\theta_{(1)})$$

since due to the compactness of $\Theta$ we can differentiate under the integral sign in (4) ( the integral of a derivative converges uniformly). From this (4) immediately follows in usual way.

Our problem can naturally be interpreted in terms of the game theory, considering that one player chooses a design maximizing $\Delta(\epsilon_0,\ \theta_{(1)})$ and the other one

chooses $\theta_{(1)},\ ||\theta_{(1)}|| = 1$, minimizing $\Delta(\epsilon_0,\ \theta_{(1)})$.
The discussion of basic notions of the game theory can be found for example in [4]. We are especially interested in the notion of a saddle pair of strategies which in our case are maximin design and the distribution $\nu$ upon the set $\Theta_1$ for which:

$$\int \Delta(\epsilon^\star,\ \theta_{(1)}) d\nu(\theta_{(1)}) = \max_{\epsilon} \min_{\mu(\Theta)=1} \int \Delta(\epsilon,\ \theta_{(1)}) d\mu = \min_{\mu(\Theta)=1} \max_{\epsilon} \int \Delta(\epsilon,\ \theta_{(1)}) d\mu. \tag{6}$$

In our case the optimal strategy in $\epsilon$ can be chosen "pure" and not mixed (the distribution $\nu(\theta_{(1)})$ as in the case of strategy in $\theta_{(1)}$) for the following reasons: First assume the first player choosing information matrix $M(\epsilon)$ of design [3]. In this case the pay-off $\Delta$ becomes a convex function of $M(\epsilon)$ on the convex set $\{M(\epsilon)|supp(\epsilon) \in X\}$ being continuous in both arguments which run over the compact subsets of finite dimensional spaces, while the set $\mathcal{M} = \{M(\epsilon)\}$ of the strategies of the first player is convex. It follows from [4] that the optimal strategy for the first player is a pure strategy $M^\star$. Representing $M^\star = M(\epsilon^\star)$ we obtain the pure maximin design. For proving that the pair $\epsilon^\star,\ \nu(\theta_{(1)})$ is a saddle pair in our case it is sufficient to verify ([4]) that:

$$\int \Delta(\epsilon,\ \theta_{(1)}) d\nu(\theta_{(1)}) \le \int \Delta(\epsilon^\star,\ \theta_{(1)}) d\nu(\theta_{(1)}) \le \Delta(\epsilon^\star,\ \theta_{(1)}) \tag{7}$$

# 3 Testing General Degree in Univariate Case.

The results of this section are preparatory for multivariate case, although some of them are of independent interest. We begin with trigonometric regression to which the algebraic case will be reduced. The principal role is played by the following simple result on the orthogonality of trigonometric functions which is easily obtained by summing up the geometrical series $\sum_{r=1}^{N} e^{irx}$.

**Lemma 3.1.**
(1) $\sum_{r=1}^{N} sin(m\frac{2\pi r}{N}) \equiv 0$, *if all* $m,\ N$ *are integers.*
*(2)* $\sum_{r=1}^{N} cos(m\frac{2\pi r}{N}) = \begin{cases} 0, & \frac{m}{N} \text{ is not an integer,} \\ N, & \frac{m}{N} \text{ is an integer.} \end{cases}$

We consider uniform $N$-design $e_N$ on the circle $\{[-\pi,\ \pi],\ \pm\pi$ are glued together$\}$, giving equal weights $\frac{1}{N}$ to all equidistant points $t_i$ ( particularly, we denote $e_N^0$ a special case of design $e_N$, having one of the points at 0).

Next lemma 3.2. follows from lemma 3.1 and a well-known trigonometric formula

**Lemma 3.2.**

(1) *System of functions:* $\sqrt{2}\ sin\ rt, \sqrt{2}\ cos\ rt, \quad r \le d$ *is orthonormal on* $e_N$ *when* $N > 2d$.

(2) *System of functions:* $1, \ \sqrt{2}\ sin\ rt, \sqrt{2} \ \ r \le d, \quad \sqrt{2} cos\ rt, \quad r < d, \quad cos\ dt$ *is orthonormal upon* $e^o_{2d}$.

Maximin design for multivariate trigonometric regression will be constructed in our last section. Next we study even trigonometric regression:

$$\eta(t,\ \theta) = \sum_{r=0}^{d} \theta_r\ cos\ rt, \qquad 0 \le t \le \pi$$

Introduce the design $\epsilon_d$ with equal weight $p_r = \frac{1}{d}$ at points $r\frac{\pi}{d}$, $r = 1, 2, \ldots, d-1$, and weights $p_0 = p_d = \frac{1}{2d}$ at points 0 and $\pi$.
Note that if the design $\epsilon_d$ is reflected across the origin and the points $\pm\pi$ are glued together, we obtain the design $e^0_{2d}$. Evenness of the $cos(\cdot)$ and point (2) of lemma 3.2. imply:

**Lemma 3.3.** *System of functions* $1, \ \sqrt{2} cos\ rt, \quad r < d, \quad cos\ dt,$ *is orthonormal on* $\epsilon_d$.

Next is

**Theorem 3.1.**
*The design* $\epsilon_d$ *minimizes* $Var(\hat{\theta}_d)$ *and the non-centrality parameter* $\Delta(\epsilon,\ \theta_d)$ *when testing the hypothesis:* $\theta_d = 0$.

**Proof.** Both statements of the theorem are equivalent since $\Delta = \frac{\theta_d^2}{D\hat{\theta}_d}$. Let us prove the second one. We have

$$\begin{aligned}
\sigma^2 \Delta(\epsilon_d,\ \theta_d) &= \min_{\tilde{\theta}_0, \ldots, \tilde{\theta}_{d-1}} \int (\eta(t,\ \theta) - \sum_{r=0}^{d-1} \tilde{\theta}_r\ cos\ rt)^2 d\epsilon_d \\
&= \min_{\tilde{\theta}_0, \ldots, \tilde{\theta}_{d-1}} \int (\sum_{r=0}^{d-1} (\theta_r - \tilde{\theta}_r)^2\ cos^2 rt + \theta_d^2\ cos^2 dt)\ d\epsilon_d
\end{aligned}$$

due to the orthogonality of the system $cos\ rt, \quad r \le d$ over $\epsilon_d$. Obviously, the minimum is attained when $\tilde{\theta}_r = \theta_r, r < d$, and equals ${\theta_d}^2$; besides,

$$\delta^2(t,\ \epsilon_d) = \tilde{\theta}_d^2 cos^2 dt$$

maximum at each point of the $\epsilon_d$'s support. The optimality of the design follows.
Now let us go over to the algebraic regression

$$\eta(x,\ \theta) = \sum_{r=0}^{d} \theta_r x^r, \qquad |x| \le 1 \tag{8}$$

Transforming the independent variable $x = cos\ t$ and plugging it in the expression

$$cos^r t = 2^{1-r} cos\ rt + \sum_{u=1}^{\lfloor \frac{r}{2} \rfloor} a_u\ cos\ (r-2u)t \tag{9}$$

where $a_u$ are some constants, gives

$$\eta(x,\ \theta) = \theta_d\ 2^{1-d} cos\ dt + \theta_{d-1}\ 2^{2-d} cos\ (d-1)t + \sum_{r=0}^{d-2} b_r cos\ rt, \tag{10}$$

where $b_r$ are certain linear combinations of parameters $\theta_r,\ r \leq d$.
The polynomial $T_r(x)$ which is uniquely defined by the condition

$$T_r(x) = cos(r\ arc\ cos\ x), \qquad |x| \leq 1, \tag{11}$$

is called $r$-th Chebyshev's polynomial. For $r = 0, 1, \ldots, k$ they constitute a basis in the linear space of polynomials of degrees up to $k$. Equation (10) is now equivalent to

$$\eta(x,\ \theta) = \theta_d\ 2^{1-d} T_d(x) + \theta_{d-1}\ 2^{2-d} T_{d-1}(x) + \sum_{r=0}^{d-2} b_r T_r(x) \tag{12}$$

Introducing the design $\zeta_d$ with equal weights $p_r = \frac{1}{d}$ at points $cos\ \frac{r\pi}{d},\quad r = 1, \ldots, d-1$ and $p_0 = p_d = \frac{1}{2d}$ at points $\pm 1$ obtained from $\epsilon_d$ by transformation $x = cost$, we can paraphrase the result of lemma 3.3 as follows:

**Lemma 3.3** *The system of polynomials* $\sqrt{2}T_r(x),\ r < d,\ T_d(x)$ *is orthonormal on* $\zeta_d$.
Now let us go over to the statement of the main result of this section.

**Theorem 3.2.**

(i) *For polynomial regression on* $[0,1]$ $\zeta_d$ *is a maximin design for testing the hypothesis* $\theta_{(1)} = (\theta_d,\ \theta_{d-1})^T = 0$ *when* $||\theta_{(1)}||^2 = \theta_d^2 + a\theta_{d-1}^2,\quad 0 < a < 2$.

*(ii)* $\zeta_d$ *minimizes* $Var(\ \hat{\theta}_d)$.

Proof of (ii) immediately follows from equation (12) and theorem 3.1.
This fact was proved in different way in [6]. Going over to the proof of the first statement and taking into account equation (12) we get:
$\Delta(\theta_{(1)},\ \zeta_d) \geq \min_{b_0,\ldots,b_{d-2}} \int (2^{1-d}\theta_d T_d(x) + 2^{2-d}\theta_{d-1}T_{d-1}(x) + \sum(b_r - \tilde{b}_r)T_r(x))^2 d\zeta_d.$
Using lemma 3.3 we get: $\Delta \geq 2^{2-2d}\theta_d^2 + 2^{3-2d}\theta_{d-1}^2 \geq 2^{2-2d}(\theta_r^2 + a\theta_{d-1}^2) = 2^{2-2d}$, the equality being attained iff $\theta_{d-1} = 0$.
In this case the function $\delta^2(x, \zeta, d) = 2^{2-2d}T_d^2(x)$ attains its maximum $2^{2-2d}T_d^2(x)$

at all points of the supp $\zeta_d$. i.e. according to lemma 2.1. for $\tilde{\theta}_{(1)} = (1,\ 0)$ we have

$$\Delta(\tilde{\theta}_{(1)},\ \epsilon) \leq \Delta(\tilde{\theta}_{(1)},\ \zeta_d) \leq \Delta(\theta_{(1)},\ \zeta_d)$$

Thus the pair $\tilde{\theta}_{(1)},\ \zeta_d$ is a saddle pair, and $\zeta_d$ is the maximin design.

**Remark.** For the hypothesis $\theta_d = 0$ the normalizing condition is equivalent to the following:

$$\max_{|x|\leq 1} |\eta(x,\ \theta) - E\ \hat{\eta}_{d-1}(x,\ \epsilon)| \geq 2^{1-d} \tag{13}$$

An interesting problem is the corresponding reformulation of our normalization $\theta_d^2 + a\theta_{d-1}^2 = 1$ in terms of the bias while approximating $\eta(x,\ \theta)$ by $\hat{\eta}_{d-2}(x)$.

# Multinomial regression on a cube.

Let $X$ be the n-dimensional unit cube: $|x_i| \leq 1,\ i = 1, \ldots, n,\ \theta_{(1)}$ be the vector of coefficients of monomials with degrees from $d - k + 1$ to $d$ for an algebraic polynomial $\eta(x,\ \theta)$ and $||\theta_{(1)}||$ is an euclidian norm of $\theta_{(1)}$. By the direct product of designs $\epsilon_1$ and $\epsilon_2$ on the sets $X_1$ and $X_2$ we mean the direct product of the corresponding probability measures on a set $X_1 \times X_2$.

Denote by $x^\nu,\ \theta_\nu,\ |\nu|$ respectively a monomial $\prod_{i=1}^{n} x_i^{\nu_i}$, corresponding coefficient and $\sum_{i=1}^{n} \nu_i$; the vector $\theta_{(1)}$ with one non-zero component $\theta_\nu = 1$ we denote by $\delta_\nu$; specifically $\delta_{d,r}$ corresponds to the monomial $x_i^d$.

**Theorem 4.1.**

(1) *The direct product $\zeta_d^n$ of the designs $\zeta_d$ of §3 is a maximin design for testing a hypothesis $\theta_{(1)} = 0,\ k$ being equal to 1 or 2.*

(2) *matrix $D_1(\zeta_d^n)$ is diagonal, variance $d_\nu = D\hat{\theta}_\nu$ of the LSE $\hat{\theta}_\nu$ for $\theta_\nu,\ |\nu| > d - k$ is*

$$d_\nu = \delta^2 \begin{cases} 2^{2-2d}, & \nu = \delta_{d,r} \ \textit{for certain } r \\ \prod_{\nu_\alpha} 2^{1-2\nu_\alpha}, & \textit{otherwise} \end{cases}$$

(3) *Specifically when $k = 1,\ d = 2,\ D_1(\zeta_2^n) = 4\delta^2 I,\ \Delta(\zeta_2^n,\ \theta_{(1)}) = (2\delta)^{-2},\ D\hat{\theta}_0 = n+1$. Outside the diagonal of $D(\zeta_d^n)$ only $Cov(\hat{\theta}_0, \hat{\theta}_{ii})$ differ from 0, and determinant of the sub-matrix $\mathcal{D}(\zeta_2^n)$ relating to $\theta_0$ and $\theta_{\alpha\alpha},\ \alpha = 1, \ldots, n$ is minimal.*

**Remarks**

(1) For $k = 1,\ d = 2$ maximin designs have been previously found in section 2.
(2) Generalizing the theorem for normalization $\sum\limits_{|\nu|=d} \theta^2(\nu) + a \sum\limits_{|\nu|=d-1} \theta^2(\nu) = 1,\ \ 0 < a < 2$, is straightforward.

Proof of Theorem 4.1 is a natural generalization of the proof of the theorem 3.2. Using the orthogonality of the products of Chebyshev's polynomials of various variables on $\zeta_d^n$ we verify directly that $(\zeta_d^n,\ \delta_{d,r})$ is a saddle pair and statement (3).

Denote $\pi_\nu(x) = \prod\limits_{i=1}^{n} T_{\nu_i}(x_i),\ \nu_+$ is the number of $\nu_i > 0$ in the vector $\nu$.

**Lemma 4.1.** When $|\mu| \leq d,\ |\nu| \leq d$

$$\int \pi_\mu(x)\pi_\nu(x)d\zeta_d^n = \begin{cases} 0, & \mu \neq \nu \\ 1, & \mu = \nu = \delta_{d,r},\ r = 1, \ldots, n \\ 2^{-\nu_+}, & \mu = \nu \neq \delta_{d,r} \end{cases} \tag{14}$$

**Proof.** The left hand side is equal to $\prod\limits_{i=1}^{n} \int T_{\mu_i(x_i)} T_{\nu_i(x_i)} d\zeta_d(x_i)$, since $d\zeta_d^n$ is the direct product of $\zeta_d$. Hence this lemma immediately follows from lemma 3.3. Further for definiteness we consider only more complicated case $k = 2$.

**Lemma 4.2.** When $d - 1 \leq |\nu| \leq d,\ \ \delta(x, \zeta_d^n, \delta_\nu) = a_\nu \pi_\nu(x),\ a_\nu = \prod\limits_{\nu_i > 0} 2^{1-\nu_i}$

**Proof.** It is sufficient to prove that

$$\min_{\eta_{d-2}} \int (x^\nu - \eta_{d-2}(x))^2 d\zeta_d^n = a_\nu^2 \int \pi_\nu^2 d\zeta_d^n. \tag{15}$$

Transforming multipliers $x_i^{\mu_i}$ according (12), we reduce the first integrand to the following form (using the fact that degrees of monomials in the expansion of $T_n(x)$ have equal evenness):

$$x_\nu - \eta_{d-2}(x) = \prod_{\nu_i > 0} (2^{1-\nu_i} T_{\nu_i}(x_i)) + \sum_{|\mu| \leq d-2} b_\mu \pi_\mu(x)$$

with certain constants $b_\mu$. Hence according to lemma 4.1 the statement follows. Statement 2 of Theorem 4.1 follows from equation (15) and lemma 4.1.
Let us prove the diagonality of matrix $D_1(\zeta_d^n)$. For this purpose we use

**lemma 4.3.** $\Delta(\zeta_d^n,\ \theta_{(1)}) = \sum \theta_\nu^2 \Pi_{\nu_i > 0} a_\nu^2 \int \pi_\nu^2 d\zeta_d^n$

**Proof.** Using linear dependence of $\delta(x, \zeta_d^n, \theta_{(1)})$ in $\theta_{(1)}$ and lemma 4.2 we get: $\delta(x, \zeta_d^n, \theta_{(1)}) = \sum_{d-1 \le |\nu| \le d} \theta_\nu a_\nu \pi_\nu(x)$. Further

$$\begin{aligned}\Delta(\zeta_d^n,\ \theta_{(1)}) &= \int \delta^2(x, \zeta_d^n, \theta_{(1)}) d\zeta_d^n \\ &= \sum_{d-1 \le |v| \le d} \theta_\nu^2 a_\nu^2 \int \pi_\nu^2 d\zeta_d^n + \sum_{\mu \ne \nu} a_\mu a_\nu \int \pi_\mu \pi_\nu d\zeta_d^n := \Sigma_1 + \Sigma_2.\end{aligned}$$

The $\Sigma_2$ equals 0 according to lemma 4.1.

Lemmas 4.3 and 4.1, imply now point 2 of theorem 4.1, which then implies that for every $1 \le r \le n$

$$\Delta(\zeta_d^n,\ \delta_{d,r}) = \min_{||\theta_{(1)}||=1} \Delta(\zeta_d^n,\ \theta_{(1)}) \tag{16}$$

Since the supp $\zeta_d^n$ belongs to the set

$$A_r = \{x : \delta^2(x, \zeta_d^n, \delta_{d,r}) = T_r^2(x) = \Delta(\zeta_d^n,\ \delta_{d,r})\} \tag{17}$$

for any $r$, according to Theorem 2.1 (3), $\zeta_d^n$ maximizes $\Delta(\epsilon,\ \delta_{d,r})$, i.e.

$$\Delta(\epsilon,\ \delta_{d,r}) \le \Delta(\zeta_d^n,\ \delta_{d,r}) \le \Delta(\zeta_d^n,\ \theta_{(1)})$$

Hence $(\zeta_d^n,\ \delta_{d,r})$ is a saddle pair, and $\zeta_d^n$ is a maximin design. The support of $\zeta_d^n$ coincides with $\cap_{r=1}^n A_r$ implying that the support of any maximin design belongs to supp $\zeta_d^n$.

# Multinomial regression on a ball.

For the case of $d = 2,\ k = 1$ maximin design was found in section 2 with support in a ball $X = \{x : \sum x_i^2 \le 1\}$ for the same norm as in section 4. Here we generalize this result for $d > 2$ and for general norms $||\theta_{(1)}||$ which in the case of a ball it is natural to choose invariant under all rotations ("rotatable"). To be more precise, we demand that $||\theta_{(1)}||$ should coincide for polynomials $\eta_d$ and $\eta_d(\mathbf{u}x)$ where $\mathbf{u}$ is an arbitrary rotation of $X$. Examples of such norms are $\max_{x \in X} |\eta_d(x,\ \theta) - \eta_{d-1}(x,\ \mu)|$ or $\sqrt{\int (\eta_d(x,\ \theta) - \tilde{\eta}_{d-1}(x,\ \mu))^2\ d\lambda}$ where $\lambda$ and $\mu$ are certain rotatable measure upon $X$ for this general situation the following theorem is true.
**Theorem 5.1.**

(1) *The support of any maximin design for testing hypothesis $\theta_{(1)} = 0$ is contained in the spheres $\sum x_i^2 = r_i^2 \le 1$, the unit sphere included (d being even, in this formulation $\frac{1}{2}$ of a sphere is regarded as a center of the ball).*

(2) *Uniform distributions on the spheres mentioned (or their discrete approximations with same moments up to order 2d) with certain weights form maximin designs.*

**Proof.** For rotatable norms $||\theta_{(1)}||$ it follows that if $\epsilon$ is maximin, then $\mathbf{u}\epsilon$ obtained by an arbitrary rotation of $\epsilon$ is also maximin.

**Lemma 5.1.** *If $\epsilon$ is a maximin design, then $\bar{\epsilon} = \int \mathbf{u}\epsilon d\lambda(\mathbf{u})$ (where $\lambda(\mathbf{u})$ is the Haar's probability measure on the orthogonal group) is maximin.*

**Proof.** As is well known (see, e.g. [6]) $D_1^{-1}(\tilde{\epsilon}) - \int D_1^{-1}(\mathbf{u}\epsilon)d\lambda(u)$ is non-negative definite, and we have $D_1^{-1}(\mathbf{u}\epsilon) = D_1^{-1}(\epsilon)$ for all $\mathbf{u} \in U$. Thus $\theta_{(1)}^T D_1^{-1}(\tilde{\epsilon})\theta_{(1)} \geq \theta_{(1)}^T D_1^{-1}(\epsilon)\theta_{(1)}$ for all $||\theta_{(1)}||$ and, consequently $\inf\ \theta_{(1)}^T D_1^{-1}(\tilde{\epsilon})\theta_{(1)} \geq \inf\ \theta_{(1)}^T D_1^{-1}(\epsilon)\theta_{(1)}$ thus the statement of the lemma follows.

Maximin design can be chosen rotatable. Thus, it follows that the corresponding mixed minimax strategy in $\theta$ is rotatable and, consequently, is defined by certain rotatable measure $\nu$:

$$\tilde{\theta} = \int (\mathbf{u}\theta)d\nu(\theta))$$

Function $\int \delta^2(x, \tilde{\epsilon}, \nu(\theta))\ d\nu(\theta)$ of lemma 2.2. is, clearly, a function of $r^2 = ||x||^2$ only, besides this function being a polynomial of $x$ of degree $2d$ must be a polynomial $P_d$ of $r^2$ of degree $d$. The polynomial $P_d$ does not equal a constant, for its coefficient of degree $d$ cannot be (according to the definition of $\delta^2$) equal to zero. Consequently, the maximum of $P_d$ upon $X$ can be attained only on $\frac{d}{2}$ spheres and at the origin when $d$ is even and upon $\frac{d+1}{2}$ spheres when $d$ is odd.

# 4 Trigonometric regression on a torus.

We can prove stronger results here than in previous sections

**Theorem 6.1.**

(1) *For a model* (3) *of section 2 the direct product $\rho^n$ of $n$ uniform designs $e_{N_i}$ depending upon the coordinates $t_i$ is a maximin design when $N_i \geq 2d+1,\ \ i = 1, \ldots, n$. In this case*

(2) $D(\rho^n) = 2I$ and

(3) $D_1(\rho^n) = \min_{\epsilon} |D_1(\epsilon)|$.

**Proof.** Let us prove the second statement first.

**Lemma 6.1.** *The system of functions* $1,\ \sqrt{2}\ sin(\nu^T t),\ \sqrt{2}\ cos(\nu^T t)$ *is orthonormal on* $\rho^n$ *if* $N_i \geq 2d+1,\ |\nu| \leq d,\ i = 1, \ldots, n.$

**Proof**. For definiteness let us find

$$2\int sin(\mu^T t)\ sin(\nu^T t)\ d\rho^n = 2\int sin(\sum \mu_i t_i)\ sin(\sum \nu_i t_i)\ \pi\ d\epsilon_N(t_i)$$

According to the statement (1) and lemma 3.2, the integral over $t_i$ is 0 if $\mu_i \neq \nu_i$, and equals 1/2, if $\mu = \nu$, coordinates $t_j,\ j \neq i$ being fixed. Thus statement (2) follows. Further on, applying last lemma we get:

$$\sum [Var(\hat{a}_\nu) cos^2(\nu^T t) + Var(\hat{b}_\nu) sin^2(\nu^T t)] = 2\sum_A [cos^2(\nu^T t) + sin^2(\nu^T t)]$$

is the number of parameters $a_\nu,\ b_\nu,\ \nu \in A$
i.e. the condition of the truncated $D$-optimality of the design $\rho^n$ is fulfilled. Thus the proof is complete.

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