ON SEQUENTIAL DISCRIMINATION BETWEEN CLOSE MARKOV CHAINS

M.B. Malyutov[1] and D. M. Malyutov[2]

[1] Mathematics Dept., Northeastern University,Boston MA 02115 USA MLTV@neu.edu
[2] Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA USA 02139

Summary. The appropriateness of the Wald-type logarithmic asymptotics for the mean length of sequential discrimination strategies between close alternatives has been already challenged in the well-known controversy over comparative performances of the asymptotically optimal Chernoff's discrimination strategies and ad hoc heuristic rules of Box and Hill in the seventies.

We continue this discussion by showing a poor performance of the Wald-type asymptotic bounds for the mean length of asymptotically optimal sequential discrimination strategies between the simplest types of Markov chains by simulation. We propose some weak remedies against this disaster and some alternative asymptotic tools.

Keywords: Sequential test, maximal error probability, mean length of strategies, Markov chain.

1 Introduction

One of the most popular results in Wald (1947) is his logarithmic asymptotic lower bound for the mean length of sequential discrimination strategies between simple hypotheses which turned out to be asymptotically attained by his Sequential Probability Ratio Test. This asymptotic approach was generalized and extended to numerous discrimination settings between composite hypotheses and change-point problems by Chernoff, Kiefer and Sacks, Lai, and many other authors including the first author. The accuracy of the Wald-type logarithmic asymptotics for the mean length of sequential discrimination strategies between close alternatives has been already challenged in the well-known controversy over comparative performances of the asymptotically optimal Chernoff's discrimination strategies and the ad hoc heuristic rules of Box and Hill (1967) in the seventies (see, e.g. Blot and Meeter, (1973)). Under error probability α interesting in practice, the residual term $o(\log|\alpha|)$ may well exceed the principal term of the Wald-type bound, see our discussion further.

We continue this discussion by showing a poor performance of the Wald-type asymptotic bounds for the mean length of asymptotically optimal sequential discrimination strategies between simplest types of Markov chains *by simulation*. We propose some weak remedies against this disaster (section 2) and some alternative asymptotic tools. We study the performance of several interrelated sequential discrimination strategies theoretically and by statistical simulation for two particular examples.

In section 2 we outline an application of general asymptotically optimal sequential discrimination strategies proposed in Malyutov and Tsitovich (2001) (abbreviated further as MT-2001) for an illustrative example of testing the correlation significance for a first-order autoregression with small noise. Here, for a certain range of parameters we can show the attainability of our asymptotic lower bound *if we permit an initial transition period to the equilibrium equilibrium for the observed MC*. An extension of the results in section 2 to the discrimination between statistical hypotheses about a general conservative dynamical system perturbed by small noise is straightforward.

Two simplified versions of discrimination strategies between Markov Chains (MC) are introduced in section 3 and studied in section 4 by simulation for testing a regular binary random number generator versus a Markov chain with *transition probabilities very close to those in the null hypothesis*. We end up with the conclusions formulated in Section 5.

Our examples are related to the general setting in MT-2001 which we now introduce. Let X be a finite set with m_X elements, $\mathcal{P}$ be a Borel set of transition probability matrices for Markov chains on state space X satisfying the conditions formulated in the next paragraph. We denote by $p(x, y), x \in X, y \in X$, elements of the matrix $P \in \mathcal{P}$. Under the convention $0/0 := 1$ we assume that for some $C > 0$

$$\sup_{P,Q \in \mathcal{P}} \max_{x \in X, y \in X} \frac{p(x, y)}{q(x, y)} \leq C < \infty \tag{1}$$

and for every $P \in \mathcal{P}$ MC with transition probability matrix P is aperiodic and irreducible which implies the existence and uniqueness of the stationary distribution $\mu := \mu_P$ with $\mu_P(x) > 0$ for every $x \in X$. It follows from (1) that $p(x, y) = 0$ for any $P \in \mathcal{P}$ entails $q(x, y) = 0$ for all $Q \in \mathcal{P}$. Our statistical decisions are based on the log-likelihood probability ratios:

$$z(P, Q, x, y) := \ln p(x, y)/q(x, y).$$

$$I(x, P, Q) := \sum_{y \in X} p(x, y) z(P, Q, x, y)$$

is the Kullback-Leibler divergence (*cross-entropy*). The set $\mathcal{P}$ is partitioned into Borel sets $\mathcal{P}_0$, $\mathcal{P}_1$ and the indifference zone $\mathcal{P}_+ = \mathcal{P} \setminus (\mathcal{P}_1 \cup \mathcal{P}_0)$. We test $H_0 : P \in \mathcal{P}_0$ versus $H_1 : P \in \mathcal{P}_1$, every decision is good for $P \in \mathcal{P}_+$. Suppose that the divergence between the hypotheses is positive, i.e.

$$\min_{i=0,1} \inf_{P \in \mathcal{P}_i, Q \in \mathcal{P}_{1-i}} \max_{x \in X} I(x, P, Q) \geq \delta_0 > 0. \tag{2}$$

The probability law of $X_i, i = 0, 1, \ldots$, is denoted by $\mathbf{P}_P$ and the expectation is denoted by $\mathbf{E}_P$. In particular

$$I(x, P, Q) = \mathbf{E}_P(z(P, Q, X_0, X_1)|X_0 = x).$$

A *strategy* s consists of a stopping (Markov) time N and a measurable binary decision δ, $\delta = r$ means that H_r , $r = 0, 1$, is accepted. Introduce α-*strategies* s satisfying

$$\max_{r=0,1} \sup_{P \in \mathcal{P}_r} \mathbf{P}_P(\delta = 1 - r) \leq \alpha.$$

$\mathbf{E}_P^s N$ is the *mean length* (MEAL) of a strategy s. **Define**: $I(\mu, P, Q) := \sum_{x \in X} \mu(x) I(x, P, Q)$, where μ is a probability distribution on X, $I(P, Q) := I(\mu_P, P, Q)$ and $I(P, \mathcal{R}) := \inf_{Q \in \mathcal{R}} I(P, Q)$ for $\mathcal{R} \subset \mathcal{P}$; $A(P) := \mathcal{P}_{1-r}$ for $P \in \mathcal{P}_r$ as the alternative set to P (in $\mathcal{P}$). For $P \in \mathcal{P}_+$, if $I(P, \mathcal{P}_0) \leq I(P, \mathcal{P}_1)$, then $A(P) := \mathcal{P}_1$, otherwise, $A(P) := \mathcal{P}_0$. Finally, $k(P) = I(P, A(P))$. It follows from (2) that

$$k_0 := \inf_{P \in \mathcal{P}} k(P) > 0, P \in \mathcal{P}$$

since $\mu_P(x) > 0$ for all $x \in X$ and for any $P \in \mathcal{P}$. It is proved in MT-2001 that for every $P \in \mathcal{P}$ and α-strategy s

$$\mathbf{E}_P^s N \geq \rho(\alpha, P) + O\left(\sqrt{\rho(\alpha, P)}\right), \tag{3}$$

as $\alpha \to 0$, where $\rho(\alpha, P) = |\ln \alpha|/k(P)$ is the well-known principal term of the MEAL first appearing in Wald (1947), and the following α-strategy s^1 attaining equality in (3) is constructed depending on a parameter β, $\beta < \alpha$. Strategy s^1 consists of conditionally i.i.d. loops. Every loop contains two phases. Based on the first

$$N_1 = N_1(\alpha, \delta_0) \tag{4}$$

observations of a loop, we estimate the matrix P by the MLE $\hat{P} \in \mathcal{P}$ (or its orthogonal projection on $\mathcal{P}$ as a subset of $\mathbf{R}^{m_X^2}$). Let us enumerate measurements of the second phase anew and introduce $L_k(\hat{P}, Q) = \sum_{i=1}^{k} z(\hat{P}, Q, X_{i-1}, X_i)$.We stop observations of the loop at the first moment N_2 such that

$$\inf_{Q \in A(\hat{P})} L_{N_2}(\hat{P}, Q) > |\ln \beta| \tag{5}$$

or

$$N_2 > N_0 := 2k_0^{-1} |\ln \alpha|, \tag{6}$$

stop all experiments and accept the hypothesis H_r (i.e. $\delta = r$), if (5) holds and $1-r$ is the index of the set $A(\hat{P})$. After event (6) we begin a new loop. Strategies s^1 and s^2 (see section 3) do not revise the estimate $\hat{P}$ during the second phase of a loop which we modify for strategy s^3 introduced in section 3. In both new strategies the rule (5) is replaced by comparing (with the prescribed level) only the *likelihood ratios with respect to the closest alternatives* to $\hat{P}$ which is numerically much easier to implement. Our simulation in section 4 shows that s^3 is much better than s^2. Note also that for attaining an asymptotic equality in (3) it is assumed in MT-2001 that $P(N_2 > N_0) \to 0$ as $\alpha \to 0$ making the probability of more than one loop negligible. This holds, if

$$EI(P, \hat{P})/\delta_0 \to 0 \text{ as } \alpha \to 0. \tag{7}$$

We study the situation of *close alternatives (δ_0 small)* in both our examples further which is dubious to deal with the conventional *asymptotic approach of large deviations* common in discrimination problems, see e.g. Chernoff (1972). Le Cam's theory of *contiguous alternatives might give a better approximation* which we plan to study in future. Namely, the misclassification probability under the hypotheses at distance of order $cn^{-1/2}$, where n is the sample size of the first stage, can be shown to be normal with parameter depending on c. Hence we will be able to choose parameters in such a way that the unfavorable outcome of the first loop would take place with probability less than α.

In sections 3 and 4 the condition (7) is impractical since $L = |\ln \alpha|$ cannot be very large. Our simulation shows that under the parameters studied only α- strategies with several times larger MEAL than $\rho(\alpha, P)$ seem to be attainable which appears around twice less than the sample size of the best static strategy. It is an open problem whether our strategies can be modified to require the MEAL equivalent to the lower bound proved so far.

Remark. The most promising revision of our strategies would be following: generalizing to the first stage of our strategies the recent methods of supervised discrimination *maximizing margin*. These methods use the estimation of the likelihood function only in the vicinity of the *margin* points crucial for discrimination getting rid of the idea to approximate the likelihood function globally (and plug in the estimated parameters there).

In the next section 2 we study "small noise" case where condition (7) can be achieved under mild conditions in the*MC transition period*, where the *signal-noise ratio exceeds considerably that for the stationary distribution*. Even simpler is to justify the condition (7) in non-stationary "signal plus small noise" models which we do not consider here.

2 Testing correlation in a first-order autoregression

Here we illustrate the previously exposed general results for an example of sequential discrimination between $i.i.d.N(0, \varepsilon^2)$ measurements versus a first order autoregression with small correlation. We view this as an example of conservative dynamical systems perturbed by small noise which can be treated similarly.

Consider a Markov chain $X_0, X_1, \ldots$ with joint distribution P_θ:

$$X_t = \theta X_{t-1} + \varepsilon e_t, t = 1, 2, \ldots,$$

where $|\theta| \le \Theta < 1$ is an unknown correlation, the noise εe_t is $i.i.d.N(0, \varepsilon^2)$ and we can choose X_0 to be, say, 1 (or more generally is random with constant mean as $\varepsilon \to 0$).

We test $H_0 = \{\theta = 0\}$ versus the composite hypothesis $H_1 = \{|\theta| \ge d > 0\}, \{0 < |\theta| < d$ being an indifference zone. The marginal distribution of X_t is well-known to converge exponentially fast as $t \to \infty$ to $N(0, \varepsilon^2(1-\theta^2))$ for every initial state. We study here the performance of strategy s^1 for small d, ε and α. The loglikelihood of P_θ versus $P_{\dot\theta}$ up to moment T is $Z_0 + \sum_1^{T+1} Z_t$, where

$$Z_t := [(X_t - \dot\theta X_{t-1})^2 - (X_t - \theta X_{t-1})^2]/(2\varepsilon^2).$$

First averaging Z_t given X_{t-1}, we get

$$I(x, \theta, \dot\theta) = (\theta - \dot\theta)^2 x^2/(2\varepsilon^2),$$

and then averaging over the stationary distribution, we get the stationary cross-entropy

$$I(\theta, \dot\theta) := \mathbf{E}_\theta(Z_t) = (\theta - \dot\theta)^2(1-\theta^2).)$$

In particular, $I(0,\theta) = \theta^2, I(\theta, 0) = \theta^2(1-\theta^2)$. It is straightforward from the above calculations that the Fisher information $J(x, \theta)$ of X_t given that $X_{t-1} = x$ is $x^2/(2\varepsilon^2)$. Thus

$$E(\sum_0^T J(X_t)) = \varepsilon^{-2} \sum_{t=0}^T (\theta^{2t})/2$$

is not less asymptotically for large T than $1/[2(1-D^2)\varepsilon^2]$ implying that the variance of the preliminary MLE $\hat\theta$ based on $\sqrt{\rho(\alpha, P_d)}$ observations is $1/\sqrt{\rho(\alpha, P_d)}$, if we assume that $\varepsilon^2(1-D^2) = o(d^2/L)$.

This implies the attainment of the lower bound (3) by s^1 along the lines of MT-2001. The bound (3) holds, if the transition period to the stationary distribution is not sufficient to discriminate with error probability less than α, which is also straightforward to rephrase in terms of the model parameters.

3 Testing Random Number generator vs. Markov chain

Our basic hypothesis H_0 deals with a Bernoulli binary equally likely distributed (P_0) sequence of measurements $X_1, X_2, \ldots$. We test it versus an alternative hypothesis H_1 that the sequence observed is a stationary Markov chain with transition probabilities $P_r := (p_{ij}, i, j = 1, 2)$, where $r := (r_1, r_2), r_1 := p_{11} - 1/2, r_2 := p_{22} - 1/2$, such that for certain $d > 0$

$$I(P_0, P_r) := -\ln[16 p_{11}(1-p_{11}) p_{22}(1-p_{22})]/4 \ge d^2.$$

Note that $I(P_0, P_r) = ||r||_2^2(1 + o(1))$ as $||r||_2 \to 0$, where

$$||r||_2^2 := r_1^2 + r_2^2.$$

For very small d considered in our simulation, we can approximately view the set of alternatives as the exterior to the figure which is very close to a circle of radius d with center in $(1/2, 1/2)$. We skip a more lengthy general expression for $I(P_r, P_{\dot r})$ unused in our presentation of the simulation results. The local optimization method used in our program for finding the set of P_{r^*} minimizing $I(P_{\hat r}, P_{\dot r})$ over $P_{\dot r}$ on the border of the alternative set to a preliminary estimate $P_{\hat r}$, always unexpectedly found a *unique* minimizing point $A(\hat r)$ (which simplifies the strategy considerably). We are not aware if this is true in general situations. Now we introduce two sequential algorithms for the discrimination between H_0 and H_1. The only difference of the strategy s^2 from s^1 is that the rule (3) is replaced with

$$L_{N_2}(P_{\hat{r}}, P_{A_{\hat{r}}}) > |\ln \alpha|. \tag{8}$$

To simplify the strategy further, we abandon another parameter $\beta < \alpha$ in the definition of s^1 which apparently does not change the performance of s^3 considerably (see the corresponding figures further) whereas the evaluation of the recommended in MT-2001 update is clumsy.

Now, the strategy s^3 deviates from s^2 only in being more greedy: we continue to update recurrently the preliminary estimate for the true parameter r during the second phase in parallel to counting the likelihood ratios, and if a loop ended undecidedly, we plug the updated estimate for r into the likelihhod ratio, find the closest alternative, and start the new second phase. Therefore, only second phases occur in loops after the first one.

3.1 Static Discrimination

Now we discuss the performance of the best static (non-sequential) strategy with maximal error probability α. Our large sample discrimination problem is obviously asymptotically equivalent to the discrimination of the zero mean hypothesis for the bivariate rotationally invariant Normal distribution versus the spherically invariant alternative dealt with in Example 5.17 of Cox and Hinkley (1974). It is shown there that the best critical region is the exterior to a circle of certain radius, and that the distribution under the alternative is the non-central Chi-Square with two degrees of freedom. Using the power diagrams of non-central Chi-Square in Sheffe (1958), we find that the radius providing the equality of maximal errors under the null hypothesis and the alternative is approximately $0.39d$. This finally determines the sample size such that the maximal error probabilities equal specified levels which we compare with the empirical MEALs of strategies s^3, s^2 found by simulation.

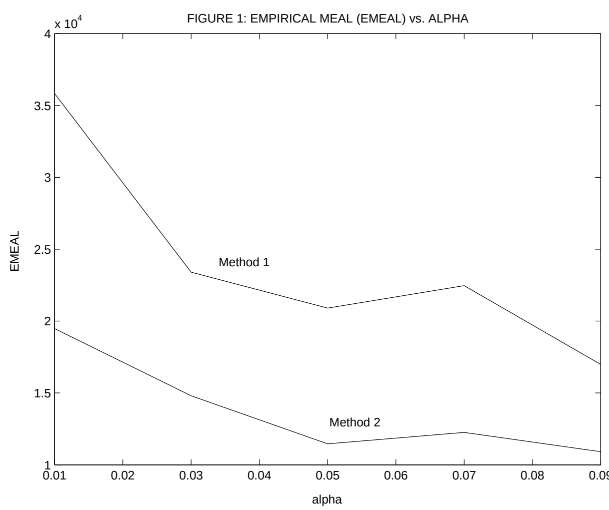

Figure 1. Empirical MEAL (EMEAL) VS. α

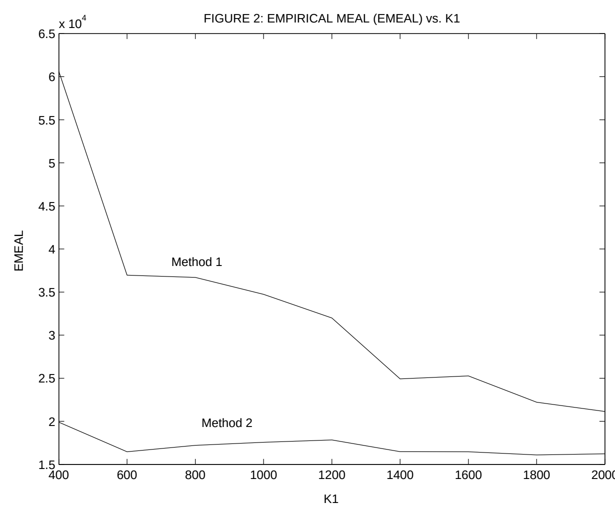

Figure 2. Empirical MEAL (EMEAL) VS. K1

4 Simulation Results

We present a series of simulation results (using MATLAB) for s^2 and s^3 with various parameters of the model described in the preceding section. The code is available by request.

The table below summarizes the results of a few trials of our simulation. In the table, $N_1 = K_1\sqrt{L}/d^2$, where $L = |\ln(\alpha)|, n$ is the number of times strategies were run under the same parameters, the empirical MEAL (EMEAL) is the average number of random numbers taken before the decision, the ENOL is the average number of loops over n runs. Note that all the parameters in simulations 2-4 are taken the same with n=100, and with $n = 1000$ in simulation 5 for estimating the variance of the performance parameters.

		d^2	α	K_1	p	n	EMEAL	FE	ENOL
(1)	s^2	0.0002	0.01	2500	0.5015	200	263723.82	0.02	4.37
	s^3						93836.39	0.0	1.905
(2)	s^2	0.001	0.02	500	0.5	100	42434.89	0.09	4.27
	s^3						15553.15	0.03	1.89
(3)	s^2	0.001	0.02	500	0.5	100	49191.76	0.04	4.8
	s^3						15879.13	0.01	1.97
(4)	s^2	0.001	0.02	500	0.5	100	52232.04	0.04	4.97
	s^3						16412.41	0.03	1.98
(5)	s^2	0.001	0.02	500	0.5	1000	42934.085	0.057	4.24
	s^3						15729.03	0.01	1.90

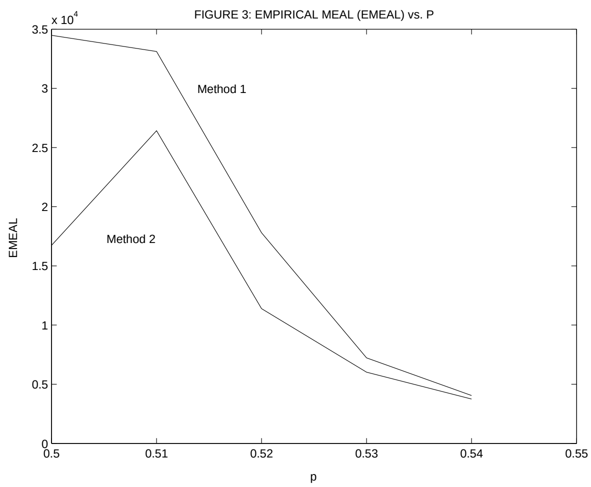

Figure 3. Empirical MEAL (EMEAL) VS. P

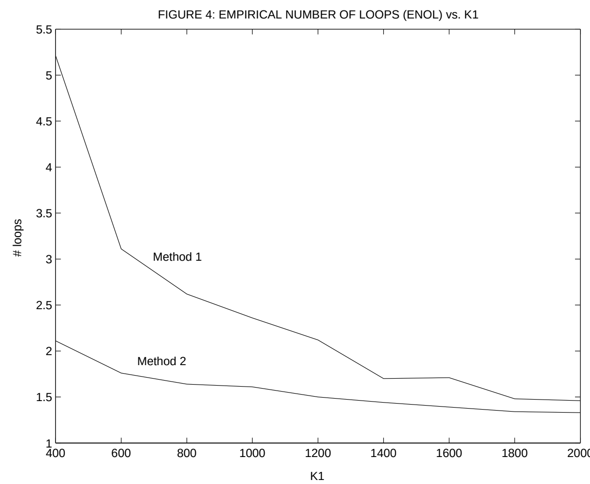

Figure 4. Empirical number of loops (ENOL) VS. K1

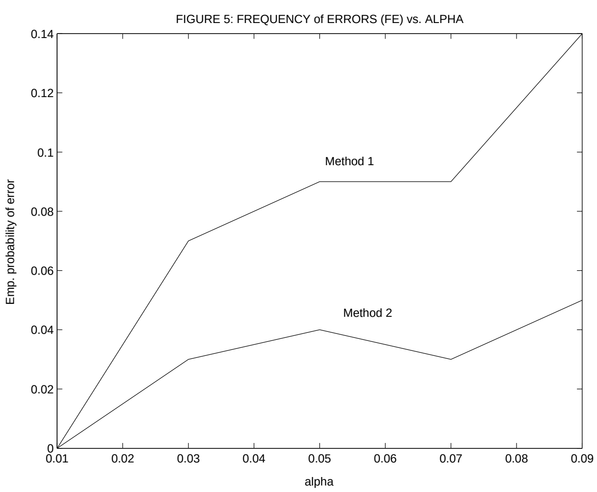

Figure 5. Frequency of errors (FE) VS. α

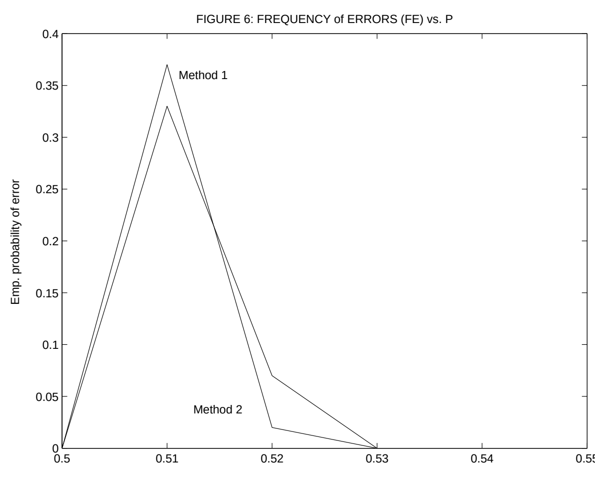

Figure 5. Frequency of errors (FE) VS. P

In the plots $d^2 = 0.001, p := p_{11} = p_{22}$, methods 1 and 2 represent respectively s^2 and s^3. Figures 3 and 6 plot respectively EMEAL and FE under various alternatives. Both are maximal, as natural, in the middle of the indifference zone. Other figures illustrate the performance of our strategies under H_0. The empirical MEAL, number of loops and empirical error rate are plotted versus changing values of various parameters of our strategies. The main news is that the EMEAL exceeds the theoretical principal term approximately four times under best parameters of our strategies.

5 Conclusions

1. Conventional asymptotic expansions of the MEAL in terms of $\ln \alpha$, where α is the maximal error probability can be of dubious importance for discrimination between close hypotheses.
2. Our simulation shows that strategy s^3 which keeps updating the preliminary estimate of the true parameter is clearly preferable as compared to the theoretically justified strategy s^1.
3. Further work to find a valid expansion for the MEAL of discriminating between close hypotheses seems necessary incorporating Le Cam's contiguity techniques.
4. Although seemingly suboptimal, the strategy s^3 is clearly preferable to the best static discrimination strategies even for discrimination between close hypotheses.
5. Use of the MC transition periods for preliminary estimation of true parameters may be fruitful in discrimination between almost deterministic ergodic Markov chains.
6. Simplified versions of the best controlled and of the change-point detection strategies, also studied in MT-2001, as well as the development of s^1, and of strategies in Lai (1998) should also be examined by simulation.

References

[Blot, W.J. and Meeter, D.A.] 1973). *Sequential Experimental Design Procedures*, Journal of the American Statistical Association, **68**, 343, 586-593.

[Box, G.E.P. and Hill, W.J.] 1967). *Discrimination between Mechanistic Models*, Technometrics, **9**, 57-71.

[Chernoff, H.] 1972). *Sequential Analysis and Optimal Design*. SIAM, Philadelphia.

Cox, D.R. and Hinkley, D.V. (1974). *Theoretical Statistics*.Chapman and Hall, London.

[Lai, T.L.] 1998). *Information Bounds and Quick Detection of Parameter Changes in Stochastic Systems*, IEEE Transactions on Information Theory, **44**, No 7, p. 2917–2929.

[Malyutov, M.B. and Tsitovich, I.I.] 2001). Second Order Optimal Sequential Discrimination between Markov Chains. *Mathematical Methods of Statist.*, **10**, No. 4, Allerton Press.

[Sheffe, H.] 1958). *Analysis of Variance*. Wiley, N.Y.

[Wald, A.] 1947). *Sequential Analysis*.Wiley, New-York.