Estimation from noisy images with the EM-algorithm

Andrey Nikiforov, Mikhail Malioutov
Northeastern University, Boston, Massachusetts. *

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Abstract

The paper deals with the case when a series of noisy images is generated from a moving object or several objects. An estimation procedure is proposed that is based on the EM-algorithm. This procedure provides estimates of position, shape of the objects, and other parameters in applications.

1 Introduction

The problem discussed in this letter stems from the study of motion of band 3 protein molecule on the red blood cell membrane. Characterization of this motion is of large biological interest. A gold or latex bead of size from 5 to 40 nm is attached to the molecule, and plane images (typically 128 by 80 pixels) are recorded at speed up to 10,000 per second to track molecules. Optical microscopy for such objects that are smaller than the resolution limit of visible light provides images of beads ("bead spots"), that are often very noisy, have complex shape and are of sizes from 100 to 500 nm, that exceeds the desired accuracy of the bead position estimate by order.

Heuristical thresholding or moment-based algorithms used by biologists, besides being inaccurate in strong noise, are heavily biased when bead spots overlap and/or when a spot is too large or bead is too close to the image edge to fit a spot into the image frame due to the limited view area of the microscope.

We propose a procedure based on the EM-algorithm that estimates the bead position more accurately, and is optimal in a certain sense. The main idea of the approach is to estimate parameters relevant to the bead spot generation using a series of images, and then to solve the inverse problem by establishing the most probable center coordinate for each image along with other characteristics (variance etc).

* *Address for correspondence:* 101, Eliot str., Chestnut Hill, MA 02467, USA. *E-mail:* mmltv@usa.net, nikf@usa.net.

The approach described is universal concerning problems of estimating object parameters like position or shape from series of noisy images or signals, when it is not not possible to point out these parameters explicitly and "self-learning" is required.

2 Model

Consider the case of a single bead on the image. Let $i = (i_x, i_y)$ be bivariate integer index on the rectangular grid R with the diagonal $(1,1) - (I_x, I_y)$. The bead induces the intensity field $h(i|a)$ on R that is parametrised by a vector parameter a. Let $a = (m, d, c)$, where $m = (m_x, m_y)$ is a bead "center", and m_x, m_y, d, c are positive real numbers. The function

$$h(i|a) = max\{0, c - ||i - m||^2/d^2\} \tag{1}$$

is used in the sequel. Here $||r||$ stands for the Euclidian norm of the vector r.

We observe a series of images at moments $t = 1, \ldots, T$. The intensity in the point i at the moment t is assumed to be

$$z_t(i) = h(i|a_t) + \varepsilon_t(i), \tag{2}$$

where a_t is the random parameter value at the moment t, supposed to be distributed independently for all t with density function $g(a|\theta)$, θ being unknown parameter of this distribution. Let $Z_t = (z_t(1,1), \ldots, z_t(I_x, I_y))$ denote the intensity field for the whole image at the moment t.

To specify the model completely, suppose that $\varepsilon_t(i)$ is a random variable distributed normally $N(0, \sigma^2)$ and independently for all t and i. Let $n(x|\sigma)$ be the respective density function. Denote by $\phi = (\theta, \sigma)$ the complete parameter set.

3 Problem

Given the set $Z_1^T = (Z_1, \ldots, Z_T)$ of T images, we should evaluate the bead position for each t. In terms of (1), values of m_t, $t = 1, \ldots, T$, are to be defined. It is also instrumental to get variance of the bead position estimate.

For simplicity we suppose first that intensity function $h(i|a)$ has a finite support $S(a)$ (this is obviosly true for (1)) and that for every t we have $S(a_t) \in R$, i.e., each image contains the bead spot completely.

4 Solution

We solve this problem in two steps. At the first step we estimate the parameter ϕ using the EM-algorithm (Dempster et. al., 1977). This is a statistical tool designed for maximum likelihood estimation when the data observed X_{obs} can be described as a part of a complete data set $X_{complete} = (X_{obs}, X_{mis})$, where X_{mis} is, at least formally, missing data subset.

For the problem specified, let us treat Z_k as observed data, and a_k as missing data. Therefore, the pair (Z_k, a_k) is the complete observation at moment t.

We use the superscripts "-" and "+" of ϕ for the current and next iterations respectively. The general description of the EM-algorithm is as follows:

- E–step. Determine $Q(\phi|\phi^-)$, where

$$Q(\phi|\phi^-) = E\left(\log f(x_{complete}|\,\phi)|\, x_{observed}, \phi^-\right).$$

  Here E stands for conditional expectation over missing part of data, given $x_{observed}$ and ϕ^-.

- M–step. Determine $\phi^- = arg\max_\phi Q(\phi|\,\phi^-)$.

Following this general description and the treatment of the model accepted, we derive the EM-algorithm for our problem.

E–step. Due to independence of random variables $e_k(i)$, we get:

$$f(x_{complete}|\,\phi) = \prod_{k=1}^{T}\Big(g(a_k|\theta)\prod_i n(z_k(i) - h(i|a_k)\mid\sigma)\Big), \qquad (3)$$

where summation and multiplication over i hereafter denote the respective operations over all elements of the rectangular grid R, and

$$Q(\phi|\phi^-) =$$
$$\sum_k \int p(a_k|Z_k,\phi^-)\Big(log\; g(a_k|\theta) - \sum_i\Big[log(\sqrt{2\pi}\sigma) + (z_k(i) - h(i|a_k))^2/(2\sigma^2)\Big]\Big)da_k,$$

where $p(a|Z_k,\phi^-)$ is the posterior probability of a to be observed, given Z_k and parameter value at the previous iteration. This probability is calculated from the Bayes formula:

$$p(a|Z_k,\phi) = \frac{p(Z_k|a,\sigma)g(a|\theta)}{p(Z_k|\phi)}, \qquad (4)$$

where the denominator is the normalizing function and $p(Z_k|a,\sigma)$ in our model is

$$p(Z_k|a,\sigma) = \prod_i n(z_k(i) - h(i|a)\mid\sigma),$$

M–step. Maximizing $Q(\phi|\phi^-)$ over ϕ, we get:

$$(\sigma^2)^- = \frac{1}{I_x I_y T}\sum_k \int p(a|Z_k,\phi^-)\sum_i (z_k(i) - h(i|a))^2 da, \qquad (5)$$

$$\theta^{-} = arg\ max_{\theta} \sum_{k} \int p(a|Z_k, \phi^{-})\ log\ g(a|\theta) da. \tag{6}$$

Finally, having the maximum likelihood estimate $\hat{\phi}$ from the EM-algorithm, (4) is the conditional density of a given Z_k, whence we can evaluate m for each t by, say, mean or bivariate median of the distribution $p(a|Z_k, \hat{\phi})$, according to the goals of subsequent analysis. Moreover, this density function provides variance or any other reasonable statistics describing accuracy of our evaluation.

5 Extensions

The EM-algorithm lets successfully handle more complex cases. First, we discuss a modification required to handle observations of multiple (say, $K > 1$) beads on an image. Physics of the experiments provides nonlinear generalization of single-bead model (2). The model

$$z_k(i) = H(i|A_k) + \varepsilon_k(i) \tag{7}$$

is adequate, where $A_k = (a_k^1, \dots, a_k^K)$, a_k^k is the intensity function parameter for the k-th bead, and $H(i|A_k) = max\{h(i|a_k^1), \dots, h(i|a_k^K)\}$. Generalized formulation of the problem as one involving incomplete data is obvious. The parameter set is $\phi = (\theta_1, \dots, \theta_K, \sigma)$. Assuming independence of random vectors a_k^k in both t and k, we should replace equation (3) by

$$f(x_{complete} \,|\, \phi) = \prod_k \Big(\prod_k g(a_k^k|\theta^k) \prod_i n(z_k(i) - H(i|A_k)|\sigma) \Big),$$

and, further, (5) by

$$(\sigma^2)^{-} = \frac{1}{I_x I_y T} \sum_k \int \cdots \int p(A|Z_k, \phi^{-}) \sum_i (z_k(i) - H(i|A))^2 da^1 \dots da^k, \tag{8}$$

and (6) by

$$\theta^{-} = arg\ max_{\theta} \sum_k \int \cdots \int p(A|Z_k, \phi^{-}) \sum_k log\ g(a_k|\theta_k) da^1 \dots da^k, \tag{9}$$

where $\theta = (\theta_1, \dots, \theta_K)$ and $p(A|Z_k, \phi)$ is an obvious generalization of (4). Note, that (9) factorizes into K separate equations, and (8) invokes multiple intergation only for those beads that have overlapping spots. Two-beads overlapping is observed at only 10% of images, and three-beads overlapping is extremely rare.

Next, use of the EM-algorithm for truncated data is common, and the treatment of the case when bead spots do not fit into the image frame is simple: include unobserved part of the intensity field into missing data component x_{mis}. We omit the respective modification description.

Finally, modelling function (1) is oversimplification of the intensity field. More complex models can be used in our algorithm. Nevertheless, (1) provides good fit in a small area around the bead center that was good in our analysis.

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References

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