NOTES ON TANNAKIAN CATEGORIES

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Abstract. These are notes for an expository talk at the course "Differential Equations and Quantum Groups" given by Prof. Valerio Toledano Laredo at Northeastern University, Spring 2016. We give a quick introduction to the Tannakian formalism for algebraic groups.

The purpose of these notes if to give a quick introduction to the Tannakian formalism for algebraic groups, as developed by Deligne in [D]. This is a very powerful technique. For example, it is a cornerstone in the proof of geometric Satake by Mirković-Vilonen that gives an equivalence of tensor categories between the category of perverse sheaves on the affine Grassmannian of a reductive algebraic group G and the category of representations of the Langlands dual $\widehat{G}$, see [MV].

Our plan is as follows. First, in Section 1 we give a rather quick introduction to representations of algebraic groups, following [J]. The first main result of the theory says that we can completely recover an algebraic group by its category of representations, see Theorem 1.12. After that, in Section 2 we formalize some properties of the category of representations of an algebraic group G into the notion of a Tannakian category. The second main result says that every Tannakian category is equivalent to the category of representations of an appropriate algebraic group. Our first definition of a Tannakian category, however, involves the existence of a functor that may not be easy to construct. In Section 3 we give a result, due to Deligne, that gives a more intrinsic characterization of Tannakian categories in linear-algebraic terms. At the beginning of each section, its content is described in more detail.

A few words about notation. Throughout, we fix a field $\mathbb{K}$ that we do *not* assume to be algebraically closed (we do, however, assume it has zero characteristic.) Unless otherwise explicitly stated, by an *algebra* we always mean a *commutative* $\mathbb{K}$-algebra. All the categories we consider below are $\mathbb{K}$-linear.

1. Algebraic groups and their representations

We give a quick overview of some basic facts of the representation theory of algebraic groups. We remark that we take the functorial approach to algebraic geometry. By an *affine scheme* we mean a functor $X : \mathbb{K}\text{-alg} \to \mathfrak{Sets}$ that is represented by a finitely generated algebra A, that is, $X(A') = \mathrm{Hom}_{\mathbb{K}\text{-alg}}(A, A')$ for every $A' \in \mathbb{K}$-alg. If this is the case, then we denote $\mathbb{K}[X] := A$, and $\mathrm{Spec}(A) := X$. We will make extensive use of the following well-known result.

Lemma 1.1 (Yoneda Lemma)**.** *Let A be a algebra. Then, for any functor $X : \mathbb{K}\text{-alg} \to \mathfrak{Sets}$, there is a bijection* $\mathrm{Mor}(\mathrm{Spec}(A), X) \xrightarrow{\cong} X(A)$. *This bijection is given by* $\Phi \mapsto \Phi_A(1_A)$.

Let us name a few consequences of the Yoneda Lemma. First, for two algebras A and A' we get:

$$\mathrm{Mor}(\mathrm{Spec}(A), \mathrm{Spec}(A)') \xrightarrow{\cong} \mathrm{Spec}(A')(A) = \mathrm{Hom}_{\mathbb{K}\text{-alg}}(A', A)$$

Second, let us denote $\mathbb{A}^1 := \mathrm{Spec}(\mathbb{K}[x])$. In particular, $\mathbb{A}^1(A) = A$ for any algebra A. Thus, we get:

$$\mathrm{Mor}(\mathrm{Spec}(A), \mathbb{A}^1) \xrightarrow{\cong} A \tag{1}$$

We remark that for any functor $X : \mathbb{K}\text{-alg} \to \mathfrak{Sets}$, the space $\mathrm{Mor}(X, \mathbb{A}^1)$ naturally acquires an algebra structure. For $\Phi, \Psi \in \mathrm{Mor}(X, \mathbb{A}^1)$, define $(\Phi + \Psi)_A(x) := \Phi_A(x) + \Psi_A(x)$, $(\Phi\Psi)_A(x) = \Phi_A(x)\Psi_A(x)$, where $A \in \mathbb{K}$-alg, $x \in X(A)$. With this structure, the isomorphism (1) becomes an *algebra* isomorphism. In other words, given an affine scheme X, we can always recover $\mathbb{K}[X]$ via $\mathbb{K}[X] \cong \mathrm{Mor}(X, \mathbb{A}^1)$.

This section is organized as follows. First, we recall the definition of an algebraic group and give several examples. We remark that the definition of an algebraic group we take here is not the most general one - strictly speaking, we are only working with *affine* algebraic groups- but this will be enough for us. Then, we explore connections between algebraic groups and Hopf algebras - as it turns out, the category of algebraic groups is anti-equivalent to the category of commutative Hopf algebras. In Subsection 1.3 we define representations of an algebraic group G, study a few operations on them and explore the connections between representations of G and comodules over the corresponding Hopf algebra. Finally, in Subsection 1.4 we explain how to recover an algebraic group G from its category of representations.

1.1. **Algebraic groups: Definition and examples.**

Definition 1.2. *A* $\mathbb{K}$-group functor *is a functor* $G : \mathbb{K}\text{-alg} \to \mathfrak{Groups}$. *A* $\mathbb{K}$*-group functor* G *is said to be an* algebraic group *if the composition of* G *with the forgetful functor that forgets the group structure is an affine scheme, that is, if* $G \cong \mathrm{Spec}(A)$ *for a finitely generated algebra* A. *We say that the* $\mathbb{K}$*-group functor* G *is a* pro-algebraic group *if* $G \cong \mathrm{Spec}(A)$ *for a* not necessarily finitely generated *algebra* A.

The usual constructions with groups extend to $\mathbb{K}$-group functors. For example, let G, H be $\mathbb{K}$-group functors. We denote $\mathrm{Hom}(G, H) := \{\Phi \in \mathrm{Mor}(G, H) : \Phi_A$ is a group homomorphism for any algebra $A\}$. For $\Phi \in \mathrm{Hom}(G, H)$, $\ker(\Phi)(A) := \ker(\Phi_A)$ is a $\mathbb{K}$-group functor, and it is a subfunctor of G. If, moreover, G, H are (pro-)algebraic groups, then $\ker(\Phi)$ is a (pro-)algebraic group, as follows. Consider $H(\mathbb{K}) = \mathrm{Hom}_{\mathbb{K}\text{-alg}}(\mathbb{K}[H], \mathbb{K})$. This has a unit element $1 \in H(\mathbb{K})$, that corresponds to an algebra homomorphism $1_\star : \mathbb{K}[H] \to \mathbb{K}$. Consider $\ker(1_\star)$, this is an ideal of $\mathbb{K}[H]$. Then, $\mathbb{K}[\ker(\Phi)] = \mathbb{K}[G]/(\Phi^\star(\ker(1_\star)))$.

Let us give some examples of algebraic groups.

(1) $\mathbb{G}_a$. For an algebra A, define $\mathbb{G}_a(A) := (A, +)$. This is clearly a $\mathbb{K}$-group functor. It is an algebraic group, with $\mathbb{K}[\mathbb{G}_a] = \mathbb{K}[x]$.
(2) $\mathbb{G}_m$. Now define $\mathbb{G}_m(A) := (A^\times, \times)$, the group of units of A. This is an algebraic group, with $\mathbb{K}[\mathbb{G}_m] = \mathbb{K}[x^{\pm 1}]$.
(3) V_a. Let V be a finite dimensional vector space. For an algebra A, define $V_a(A) := (V \otimes_\mathbb{K} A, +)$. The algebraic group structure on V_a is provided by $\mathbb{K}[V_a] = \mathrm{Sym}(V)$, the symmetric algebra of V. We remark that taking V to be infinite dimensional provides an example of an strictly pro-algebraic group.
(4) $\mathrm{End}(V)$. Again, let V be a finite dimensional vector space. Define $\mathrm{End}(V)(A) := (\mathrm{End}_A(V \otimes_\mathbb{K} A), +)$. This is an algebraic group and, in fact, $\mathrm{End}(V) = (V \otimes_\mathbb{K} V^*)_a$.
(5) GL_V. Now define $\mathrm{GL}_V(A)$ to be the set of all invertible endomorphisms of the A-module $V \otimes_\mathbb{K} A$, with the composition as the product. This is an algebraic group with $\mathbb{K}[\mathrm{GL}(V)] = \mathrm{Sym}(V)[\det^{-1}]$.
(6) $\mathrm{SL}(V)$. We remark that we have a morphism of algebraic groups $\mathrm{GL}_V \to \mathbb{G}_m$ that is given by the determinant. Its kernel is denoted by SL_V. Of course, $\mathrm{SL}_V(A)$ consists of all endomorphisms of the A-module $V \otimes_\mathbb{K} A$ with determinant 1.
(7) μ_n. Let $n \in \mathbb{Z}_{>0}$. For an algebra A define $\mu_n(A) := \{a \in A : a^n = 1\}$. This is an algebraic group, with $\mathbb{K}[\mu_n] = \mathbb{K}[x]/(x^n - 1)$. Note that μ_n may also be defined as the kernel of the endomorphism $\bullet^n : \mathbb{G}_m \to \mathbb{G}_m$.

1.2. **Algebraic groups and Hopf algebras.**

Let G be a (pro-)algebraic group. We remark that we have morphisms:

$$
\begin{array}{ccc}
m : G \times G \to G & \iota : G \to G & 1_G : \mathrm{Spec}\,\mathbb{K} \to G \\
m_A : G(A) \times G(A) \to G(A) & \iota_A : G(A) \to G(A) & (1_G)_A : \mathrm{Spec}(\mathbb{K})(A) = \{u\} \to G(A) \\
(x, y) \mapsto xy & x \mapsto x^{-1} & u \mapsto 1_{G(A)}
\end{array}
$$

That induce the corresponding comorphisms:

$$\Delta : \mathbb{K}[G] \to \mathbb{K}[G] \otimes_\mathbb{K} \mathbb{K}[G] \qquad S : \mathbb{K}[G] \to \mathbb{K}[G] \qquad \epsilon : \mathbb{K}[G] \to \mathbb{K}$$

which we call *comultiplication*, *antipode* and *counit*, respectively. For example, if $\Delta(f) = \sum_i f_i^1 \otimes_\mathbb{K} f_i^2$ for some $f \in \mathbb{K}[G] = \mathrm{Mor}(G, \mathbb{A}^1)$, then for every algebra A, $g_1, g_2 \in G(A) = \mathrm{Hom}(\mathbb{K}[G], A)$ we have $g_1 g_2(f) = \sum_i g_1(f_i^1) g_2(f_i^2)$. Similarly, $g^{-1}(f) = g(S(f))$ and $\epsilon(f) = 1(f)$. These formulas, together with the group actions for G imply.

Lemma 1.3. *Let G be a (pro-)algebraic group. Then, $\mathbb{K}[G]$ is a commutative Hopf algebra.*

Example 1.4. (1) *Take the algebraic group $\mathbb{G}_a$. The Hopf algebra structure on $\mathbb{K}[x]$ is given by $\epsilon(x) = 0, \Delta(x) = x \otimes_{\mathbb{K}} 1 + 1 \otimes_{\mathbb{K}} x$ and $S(x) = -x$.*
(2) *For the algebraic group $\mathbb{G}_m$, the Hopf algebra structure on $\mathbb{K}[x^{\pm 1}]$ is given by $\epsilon(x) = 1, \Delta(x) = x \otimes_{\mathbb{K}} x$, and $S(x) = x^{-1}$.*
(3) *Let $V = \mathbb{K}^n$, and denote $\mathrm{GL}_n := \mathrm{GL}_V$. Let $T_{ij} \in \mathbb{K}[\mathrm{GL}_n]$ be given by $T_{ij}(x) = x_{ij}$ for $x \in \mathrm{GL}_n(A) = \mathrm{Aut}(\mathbb{K}^n \otimes_{\mathbb{K}} A) = \mathrm{Aut}(A^n) = \{y \in \mathrm{Mat}_n(A) : y \text{ is invertible}\}$. Then, $\Delta(T_{ij}) = \sum_m T_{im} \otimes_{\mathbb{K}} T_{mj}$. The antipode is more complicated, it can be derived using Cramer's rule. Finally, $\epsilon(T_{ij}) = \delta_{ij}$.*

Now assume A is a commutative Hopf algebra. We claim that $\mathrm{Spec}(A)$ is an algebraic group. Indeed, for an algebra A' define a multiplication on $\mathrm{Spec}(A)(A') = \mathrm{Hom}_{\mathbb{K}\text{-alg}}(A, A')$ by $\mu_{A'}(f \times g)(a) = (f \otimes_{\mathbb{K}} g)\Delta(a)$. The axioms of a Hopf algebra say that this is actually a group, with the identity element being the composition $A \xrightarrow{\epsilon} \mathbb{K} \hookrightarrow A'$, and the inverse of $f \in \mathrm{Spec}(A)(A')$ being $f \circ S$. The following is now easy to show.

Lemma 1.5. *The categories of pro-algebraic groups and of commutative Hopf algebras are anti-equivalent.*

To finish this subsection, let us note that G is commutative if and only if $\mathbb{K}[G]$ is co-commutative, that is, $(12) \circ \Delta = \Delta$, where $(12) : \mathbb{K}[G] \otimes_{\mathbb{K}} \mathbb{K}[G] \to \mathbb{K}[G] \otimes_{\mathbb{K}} \mathbb{K}[G]$ permutes the tensor factors.

1.3. **Representations of algebraic groups.** Let G be a pro-algebraic group.

Definition 1.6. *A* representation *of G consists of the data (V, ρ) of a vector space V together with an algebraic group homomorphism $\rho : G \to \mathrm{GL}_V$. Note that this implies that, for every algebra A, $G(A)$ acts on $V \otimes_{\mathbb{K}} A$ by A-linear maps.*

Let us give an easy but important example of a representation of G. Let $V = \mathbb{K}$. For any algebra A, let $\rho_A : G(A) \to \mathrm{GL}_{\mathbb{K}}(A) = A^\times$ be the trivial morphism, that is, the morphism that sends $G(A)$ to 1_A. This defines a representation, called the *trivial* representation of G that we will denote simply by triv.

To avoid confusion, we will denote representations of the algebraic group G using cursive letters. That is, we denote a representation (V, ρ) simply by $\mathcal{V}$. If $\mathcal{V}$, $\mathcal{W}$ are representations of G, a *G-homomorphism* $\Phi : \mathcal{V} \to \mathcal{W}$ consists of a map $\Phi \in \mathrm{Hom}(V_a, W_a)$ such that, for every algebra A and $g \in G(A)$ the following diagram commutes:

$$\begin{tikzcd}
V \otimes_{\mathbb{K}} A \arrow[r, "\rho_A^V(g)"] \arrow[d, "\Phi_A"] & V \otimes_{\mathbb{K}} A \arrow[d, "\Phi_A"] \\
W \otimes_{\mathbb{K}} A \arrow[r, "\rho_A^W(g)"] & W \otimes_{\mathbb{K}} A
\end{tikzcd}$$

We will denote the set of G-homomorphisms from $\mathcal{V}$ to $\mathcal{W}$ simply by $\mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$. Let us see that many usual constructions with vector spaces extend to constructions with G-representations.

Tensor products. Let $\mathcal{M} = (M, \rho^M), \mathcal{N} = (N, \rho^N)$ be representations of G. Then, G acts on $M \otimes_{\mathbb{K}} N$ as follows. First, note that for any algebra A there is a natural isomorphism $M_a(A) \otimes_A N_a(A) \xrightarrow{\cong} (M \otimes_{\mathbb{K}} N)_a(A)$. The action of $G(A)$ on $M_a(A) \otimes_A N_a(A)$ is now defined diagonally: $\rho_A^{M \otimes_{\mathbb{K}} N}(g) = \rho_A^M(g) \otimes_A \rho^N(g)$. It is easy to see that this indeed gives a representation of G, that we denote by $\mathcal{M} \otimes_G \mathcal{N}$. Note that the natural $\mathbb{K}$-isomorphism $M \otimes_{\mathbb{K}} N \to N \otimes_{\mathbb{K}} M$ permuting the tensor factors gives an isomorphism $\beta_{\mathcal{M},\mathcal{N}} \in \mathrm{Hom}_G(\mathcal{M} \otimes_G \mathcal{N}, \mathcal{N} \otimes_G \mathcal{M})$. Note that $\beta_{\mathcal{N},\mathcal{M}} \beta_{\mathcal{M},\mathcal{N}} = \mathrm{id}_{\mathcal{M} \otimes_G \mathcal{N}}$.

Let us remark that this construction is functorial. For morphisms $\Phi \in \mathrm{Hom}_G(\mathcal{M}, \mathcal{M}'), \Psi \in \mathrm{Hom}_G(\mathcal{N}, \mathcal{N}')$ the morphism $\Phi \otimes_G \Psi \in \mathrm{Hom}_G(\mathcal{M} \otimes_G \mathcal{N}, \mathcal{M}' \otimes_G \mathcal{N}')$ is defined in a natural way. Let us also note that there are natural G-isomorphisms $\mathrm{triv} \otimes_G \mathcal{M}, \mathcal{M} \otimes_G \mathrm{triv} \to \mathcal{M}$, where, recall, triv is the trivial representation of G. Thus, we see that *the category of representations of G is a symmetric monoidal category.*

Duals. Now let $\mathcal{M}$ be a finite dimensional representation of G. Let us denote $M^\vee := \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$, the dual of M. Note that for every algebra A there is an isomorphism $\mathrm{Hom}_A(M_a(A), A) \xrightarrow{\cong} M_a^\vee(A)$. We may thus define an action of G on $M^\vee$ as follows. For $g \in G(A)$, $\varphi \in M_a^\vee(A)$ and $x \in M_a(A)$ we have

$(g\varphi)(x) := \varphi(g^{-1}(x))$. It is easy to see that this defines a representation of G on $M^\vee$, that we denote by $\mathcal{M}^\vee$.

Internal homs. Let $\mathcal{M}, \mathcal{N}$ be finite dimensional representations of G. Then, $\mathrm{Hom}_{\mathbb{K}}(M, N)$ admits a natural action of G, using the previous two constructions and the natural isomorphism $M^\vee \otimes_{\mathbb{K}} N \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{K}}(M, N)$. We will denote this representation by $\mathrm{Hom}_{\mathbb{K}}(\mathcal{M}, \mathcal{N})$.

Let us rephrase the notion of a G-representation in terms of the Hopf algebra $\mathbb{K}[G]$.

Definition 1.7. *Let $\mathbb{K}[G]$ be the Hopf algebra of G. A* comodule *over $\mathbb{K}[G]$ is a vector space V together with a linear map $\alpha : V \to V \otimes_{\mathbb{K}} \mathbb{K}[G]$ such that the following diagrams commute.*

$$\begin{tikzcd} V \arrow[r, "\mathrm{id}_V"] \arrow[d, "\alpha"] & V \\ V \otimes_{\mathbb{K}} \mathbb{K}[G] \arrow[r, "\mathrm{id}_V \otimes_{\mathbb{K}} \epsilon"] & V \otimes_{\mathbb{K}} \mathbb{K} \arrow[u, "\cong"] \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d, "\alpha"] & V \otimes_{\mathbb{K}} \mathbb{K}[G] \arrow[d, "\alpha \otimes_{\mathbb{K}} \mathrm{id}_{\mathbb{K}[G]}"] \\ V \otimes_{\mathbb{K}} \mathbb{K}[G] \arrow[r, "\mathrm{id}_V \otimes_{\mathbb{K}} \Delta"] & V \otimes_{\mathbb{K}} \mathbb{K}[G] \otimes_{\mathbb{K}} \mathbb{K}[G] \end{tikzcd} \tag{2}$$

As an obvious example, Δ defines a comodule structure on $\mathbb{K}[G]$. Let us see how $\mathbb{K}[G]$-comodules naturally arise from representations of G. Let $\mathcal{M}$ be a representation of G. In particular, we have an action of $G(\mathbb{K}[G]) = \mathrm{End}_{\mathbb{K}\text{-alg}}(\mathbb{K}[G])$ on $M_a(\mathbb{K}[G]) = M \otimes_{\mathbb{K}} \mathbb{K}[G]$. Consider the element $\mathrm{id}_{\mathbb{K}[G]} \in G(\mathbb{K}[G])$. Then, we may define $\alpha_M : M \to M \otimes_{\mathbb{K}} \mathbb{K}[G]$ by $\alpha_M(m) := \rho_{\mathbb{K}[G]}(\mathrm{id}_{\mathbb{K}[G]})(m \otimes_{\mathbb{K}} 1_{\mathbb{K}[G]})$. The fact that the diagrams in (2) commute simply express the fact that $1.m = m$ and that $g_1(g_2 m) = (g_1 g_2)m$ for $g_1, g_2 \in G(\mathbb{K}[G])$ and $m \in M \otimes_{\mathbb{K}} \mathbb{K}[G]$, respectively.

Now let A be any other algebra. We claim that the action of $G(A)$ on $M_a(A)$ is completely determined by the map α_M. Indeed, let $g \in G(A) = \mathrm{Hom}_{\mathbb{K}\text{-alg}}(\mathbb{K}[G], A)$. We have then a commutative diagram:

$$\begin{tikzcd} G(\mathbb{K}[G]) \times M_a(\mathbb{K}[G]) \arrow[r] \arrow[d, "G(g)\times(\mathrm{id}_M \otimes g)"'] & M_a(\mathbb{K}[G]) \arrow[d, "\mathrm{id}_M \otimes g"] \\ G(A) \times M_a(A) \arrow[r] & M_a(A) \end{tikzcd}$$

By definition, we have $g = g \circ \mathrm{id}_{\mathbb{K}[G]} = G(g)\,\mathrm{id}_{\mathbb{K}[G]}$. Thus, $g.(m \otimes 1) = (\mathrm{id}_M \otimes g) \circ \alpha_M(m)$. In other words, if $\alpha_M(m) = \sum_i m_i \otimes_{\mathbb{K}} f_i$, then

$$g.(m \otimes_{\mathbb{K}} 1) = \sum_i m_i \otimes_{\mathbb{K}} g(f_i). \tag{3}$$

By A-linearity, this completely determines the action of g on $M_a(A)$. On the other hand, if we have a $\mathbb{K}[G]$-comodule M then we can make Formula (3) a definition to get a G-representation on M. The following lemma is then easy to show.

Lemma 1.8. *Let G be a pro-algebraic group. Then, the category of representations of G is equivalent to that of $\mathbb{K}[G]$-comodules.*

Let us see a few consequences of the previous lemma. Note that if A is an algebra, M is an A-module and $m \in M$, it doesn't need to be the case that m is considered in a finite dimensional (over $\mathbb{K}$) A-module. For coalgebras, the situation is better.

Lemma 1.9. *Let M be a $\mathbb{K}[G]$-comodule and let $m \in M$. Then, m is contained in a $\mathbb{K}$-finite dimensional sub-comodule of M.*

Proof. Take a (not-necessarily finite) basis $\{c_i\}_{i \in I}$ of $\mathbb{K}[G]$. Write $\rho(m) \in M \otimes_{\mathbb{K}} \mathbb{K}[G]$ as $\rho(m) = \sum_{i \in I} m_i \otimes_{\mathbb{K}} c_i$, with only a finite number of the m_i's nonzero. We claim that the span of m and $\{m_i : i \in I, m_i \neq 0\}$ is a sub-comodule of M. This follows easily from the commutativity of the diagram in the right of (2). $\square$

Corollary 1.10. *Any finite subset of $\mathbb{K}[G]$ is contained in a Hopf subalgebra that is finitely generated as an algebra.*

Proof. Let $T \subseteq \mathbb{K}[G]$ be a finite subset. By the previous lemma, T is contained in a finite dimensional subspace M of $\mathbb{K}[G]$ with $\Delta(M) \subseteq M \otimes_{\mathbb{K}} \mathbb{K}[G]$. Let $m_1, \dots, m_n$ be a basis for M. Write $\Delta(m_i) = \sum_{j=1}^n m_j \otimes_{\mathbb{K}} a_{ij}$. It is easy to see that the subalgebra generated by $m_i, S(m_i), a_{ij}, S(a_{ij})$, $i, j = 1, \dots, n$ is a Hopf subalgebra of $\mathbb{K}[G]$. □

This immediately gives.

Corollary 1.11. *Every pro-algebraic group is the inverse limit of algebraic groups.*

1.4. **Tannakian reconstruction.** Let G be a pro-algebraic group and consider the category G-rep of finite dimensional representations of G. As we have seen, this is a symmetric, monoidal category. Note that it comes with a forgetful functor $F : G\text{-rep} \to \mathbb{K}\text{-vect}$, $\mathcal{M} \mapsto M$. We remark that for every $\mathbb{K}$-algebra A, F induces a functor $F_A : G\text{-rep} \to A\text{-mod}$, $\mathcal{M} \to M \otimes_{\mathbb{K}} A$. By definition of the tensor product $\bullet \otimes_G \bullet$, F_A is a monoidal functor for every algebra A, where A-mod is a monoidal category via $\bullet \otimes_A \bullet$. We have then an induced $\mathbb{K}$-group functor of automorphisms of F:

$$\begin{aligned}
\mathrm{Aut}^{\otimes}(F)(A) :=\ & \mathrm{Aut}^{\otimes}(F_A) \\
=\ & \{(\lambda_{\mathcal{M}})_{\mathcal{M} \in G\text{-rep}} : \lambda_{\mathcal{M}} \in \mathrm{End}_A(M \otimes_{\mathbb{K}} A), (\alpha_{\mathbb{K}} \otimes \mathrm{id}_A) \circ \lambda_{\mathcal{M}} = \lambda_{\mathcal{N}} \circ (\alpha_{\mathbb{K}} \otimes \mathrm{id}_A) \\
& \text{for every } \alpha \in \mathrm{Hom}_G(\mathcal{M}, \mathcal{N})\}
\end{aligned}$$

Note that for every algebra A and every $g \in G(A)$, we get an element $\widetilde{g} \in \mathrm{Aut}^{\otimes}(F)(A)$ as follows: for every $\mathcal{M} \in G$-rep we have that $\widetilde{g}_{\mathcal{M}}$ is simply the action of g on $M_a(A) = M \otimes_{\mathbb{K}} A$. It is easy to see that this is functorial, so that we have an homomorphism of functors $G \to \mathrm{Aut}^{\otimes}(F)$.

Theorem 1.12. *The homomorphism $G \to \mathrm{Aut}^{\otimes}(F)$ is an isomorphism.*

Before proceeding to the proof of Theorem 1.12, let us say a few words about the proof. The strategy is to first take the restriction of F to a subcategory of G-rep "generated by a single element". If $\mathcal{X}$ is such a generator, we can use results about the group GL_X. In particular, we have the following result.

Lemma 1.13. *Let M be a vector space and $G := \mathrm{GL}_M$. Let $I \subseteq \mathbb{K}[G]$ be a Hopf ideal, so that $\mathbb{K}[G]/I$ defines an algebraic group G' and we have an embedding $G' \subseteq G$. Then, there exists a finite dimensional vector space V and a line $D \subseteq V$ such that G' is the stabilizer of D on V, that is, for every $\mathbb{K}$-algebra A, $G'(A)$ is the stabilizer of $D \otimes_{\mathbb{K}} A$ on $V_a(A)$.*

Proof. Consider the regular representation $\mathbb{K}[G]$ of G. Since $\mathbb{K}[G]$ is noetherian, there exists a finite dimensional subrepresentation of $\mathbb{K}[G]$, say V', containing a generating set for I, see Lemma 1.9. In particular, G' is the stabilizer of $I \cap V'$ on V'. Let $d := \dim(I \cap V')$, $D := (I \cap V)^{\wedge d}, V := V^{\wedge d}$. We claim that G' is the stabilizer of D on V. To see this, let A be an algebra and $g \in G(A)$ be such that $g(D \otimes_{\mathbb{K}} A) = D \otimes_{\mathbb{K}} A$. Choose a basis $v_1, \dots, v_n$ of V such that:

- $v_1^A, \dots, v_m^A$ is a basis of $[(I \cap V) \otimes \mathbb{K}A] \cap g((I \cap V) \otimes_{\mathbb{K}} A)$. Here, we denote $v_i^A := v_i \otimes_{\mathbb{K}} 1_A$.
- $v_1^A, \dots, v_m^A, v_{m+1}^A, \dots, v_d^A$ is a basis of $I \cap V$.
- $v_1 A, \dots, v_m^A, v_{d+1}^A, \dots, v_{2d-m}^A$ is a basis of $g(I \cap V)$.

In particular, we have that $g^{\wedge d}(v_1^A \wedge \cdots \wedge v_d^A) = c(v_1^A \wedge \cdots \wedge v_m^A \wedge v_{d+1}^A \cdots \wedge v_{2d-m}^A)$. Since g stabilizes $D \otimes_{\mathbb{K}} A$, it follows that $\mathrm{span}\{v_1^A, \dots, v_d^A\} = \mathrm{span}\{v_1^A, \dots, v_m^A, v_{d+1}^A, \dots, v_{2d-m}^A\}$. Thus, $g(I \cap V) = I \cap V$ and $g \in G'$. We are done. □

The proof of Theorem 1.12 will now go as follows. For every representation $\mathcal{X} \in G\,$rep, we will consider the subcategory $\mathscr{C}_{\mathcal{X}}$ that is generated by $\mathcal{X}$ (see below for a formal definition). We have a map $G \to \mathrm{GL}_X$, equivalently a Hopf algebra morphism $\mathbb{K}[\mathrm{GL}_X] \to \mathbb{K}[G]$. Let us denote by $G_{\mathcal{X}}$ the subgroup of GL_X defined by the kernel of this map. By definition, this is a subgroup of the form considered in Lemma 1.13. We will see that $\mathrm{Aut}^{\otimes}(F|_{\mathscr{C}_{\mathcal{X}}})$ may be embedded in GL_X. Then, we will use Lemma 1.13 to see that, in fact, $G_{\mathcal{X}} = \mathrm{Aut}^{\otimes}(F|_{\mathscr{C}_{\mathcal{X}}})$. After that, we wil get our result by passing to the inverse limit, cf. Corollary 1.11.

Proof of Theorem 1.12. Let $\mathcal{X} \in G$-rep, and let $\mathscr{C}_{\mathcal{X}}$ be the full subcategory of G-rep that is tensor subgenerated by $\mathcal{X}$ and $\mathcal{X}^{\vee}$, that is, its objects are subquotients of objects of the form:

$$\bigoplus \mathcal{X}^{\otimes_G r_1} \otimes_G (\mathcal{X}^\vee)^{\otimes_G r_2}$$

And consider the restriction $F|_{\mathscr{C}_\mathcal{X}}$. Note that we may also interpret $\mathrm{Aut}(F|_{\mathscr{C}_\mathcal{X}})$ as a $\mathbb{K}$-group functor. Since $\mathscr{C}_\mathcal{X}$ is tensor subgenerated by $\mathcal{X}, \mathcal{X}^\vee$ we have an embedding $\mathrm{Aut}(F|_{\mathscr{C}_\mathcal{X}}) \to \mathrm{GL}_X$. On the other hand, denote by $G_\mathcal{X}$ the image of G in GL_X. We remark that the group $G_\mathcal{X}$ has the form considered in Lemma 1.13: the ideal defining $G_\mathcal{X}$ is given by the kernel of the map $\mathbb{K}[\mathrm{GL}_X] \to \mathbb{K}[G]$. We claim that the image of $\mathrm{Aut}(F|_{\mathscr{C}_\mathcal{X}})$ in GL_X coincides with $G_\mathcal{X}$. Thanks to Lemma 1.13, it is enough to check that $\mathrm{Aut}(F|_{\mathscr{C}_\mathcal{X}})$ leaves invariant every vector that is invariant under $G_\mathcal{X}$. So let $\mathscr{V} \in \mathscr{C}_\mathcal{X}$ and let $v \in V$ be invariant under $G_\mathcal{X}$. In particular, the map $\rho : \mathrm{triv} \to \mathscr{V}$, $1 \mapsto v$ is G-equivariant. Since F a tensor functor, this implies that for every automorphism λ of F and every algebra A we have:

$$\lambda_V(A)(v \otimes_\mathbb{K} 1_A) = \rho\lambda_{\mathrm{triv}}(A)(1 \otimes_\mathbb{K} 1_A) = v \otimes_\mathbb{K} 1_A,$$

So we conclude that $G_\mathcal{X} = \mathrm{Aut}(F|_{\mathscr{C}_\mathcal{X}})$. Now, if $\mathcal{X}_2$ is a subrepresentation of $\mathcal{X}_1$ then we have the following commutative diagram:

$$\begin{tikzcd} G_{\mathcal{X}_1} \arrow[r] \arrow[d] & \mathrm{Aut}(F|_{\mathscr{C}_{\mathcal{X}_1}}) \arrow[d] \\ G_{\mathcal{X}_2} \arrow[r] & \mathrm{Aut}(F|_{\mathscr{C}_{\mathcal{X}_2}}) \end{tikzcd}$$

where the vertical maps are given by restriction. Since the regular representation of G is faithful, we have that $G = \varprojlim G_\mathcal{X}$. On the other hand, it is clear that $\mathrm{Aut}(F) = \varprojlim \mathrm{Aut}(F|_{\mathscr{C}_\mathcal{X}})$. Thus, $G \cong \mathrm{Aut}(F)$. $\square$

2. Tannakian categories

In this section, we axiomatize some of the properties of the category G-rep of representations of a pro-algebraic group G. As we have seen, this category is an abelian category that comes equipped with a tensor product functor $\otimes_G : G\text{-rep} \times G\text{-rep} \to G\text{-rep}$. Moreover, we have a canonical isomorphism $\mathcal{M} \otimes_G \mathcal{N} \to \mathcal{N} \otimes_G \mathcal{M}$ that is just given by swapping the tensor factors. In other words, G-rep is a symmetric monoidal category. We use Subsection 2.1 to fix notations regarding symmetric monoidal categories. In Subsection 2.2 we introduce a class of symmetric monoidal categories in which every object has a dual object and internal homs exist, these are called *rigid* symmetric categories and G-rep is an example of them. Another important property of the category G-rep is that it admits a forgetful functor to the category of vector spaces and, moreover, we can reconstruct the group G from the category G-rep and the forgetful functor. We will axiomatize the properties of F into the notion of a *fiber functor*, see Subsection 2.3. The main result of this section says that every abelian, rigid symmetric monoidal category admitting a fiber functor is equivalent to the category of representations of an algebraic group G. We will prove this in Subsection 2.4

2.1. **Symmetric monoidal categories.**

2.1.1. *Monoidal categories.* Recall that a monoidal category $\widetilde{\mathscr{C}} := (\mathscr{C}, \otimes, \alpha, \mathbf{1}, l, r)$ is the data of:

- A category $\mathscr{C}$, that we will always assume to be $\mathbb{K}$-linear.
- A functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, called the *tensor product* of $\mathscr{C}$.
- An isomorphism of functors $\alpha : \otimes \circ (\mathrm{id}_\mathscr{C} \times \otimes) \Rightarrow \otimes \circ (\otimes \times \mathrm{id}_\mathscr{C})$, called the *associativity constraint.*
- An object $\mathbf{1} \in \mathrm{ob}(\mathscr{C})$ and isomorphisms $l : \mathrm{id}_\mathscr{C} \Rightarrow \mathbf{1} \otimes \mathrm{id}_\mathscr{C}$, $r : \mathrm{id}_\mathscr{C} \Rightarrow \mathrm{id}_\mathscr{C} \otimes \mathbf{1}$ that are called the *left* and *right unit*, respectively.

These data are supposed to satisfy the *pentagon axiom*, namely, for any objects $X, Y, Z, W \in \mathrm{ob}(\mathscr{C})$, the following diagram is commutative:

$$
\begin{tikzcd}
& W \otimes (X \otimes (Y \otimes Z)) \arrow[dl, "1_W \otimes \alpha_{X,Y,Z}"'] \arrow[dr, "\alpha_{W,X,Y\otimes Z}"] & \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[d, "\alpha_{W,X\otimes Y,Z}"'] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "\alpha_{W\otimes X,Y,Z}"] \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha_{X,Y,Z}\otimes \mathrm{id}_Z"] & & ((W \otimes X) \otimes Y) \otimes Z
\end{tikzcd}
$$

According to the MacLane-Stasheff coherence theorem, the pentagon axiom ensures that, whenever we have objects $X_1, \ldots, X_n \in \mathrm{ob}(\mathscr{C})$, any two bracketings of the expression $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ are *canonically* isomorphic.

Let us see some examples of monoidal categories.

(1) As we have seen in this class, if H is a quasi-bialgebra, then H-mod is a monoidal category.
(2) For *any* $\mathbb{K}$-linear category $\mathscr{C}$, consider the category $\mathrm{End}_{\mathbb{K}}(\mathscr{C})$ of $\mathbb{K}$-linear endofunctors of the category $\mathscr{C}$, the morphisms in $\mathrm{End}_{\mathbb{K}}(\mathscr{C})$ are given by natural transformations. Setting $\otimes = \circ$, the composition of functors; α the trivial associativity isomorphism $\alpha_{F,G,H} : (F \circ G) \circ H \to F \circ (G \circ H)$; $\mathbf{1} = \mathrm{id}_{\mathscr{C}}$ and l, r the trivial isomorphisms we get that $\mathrm{End}_{\mathbb{K}}(\mathscr{C})$ is a monoidal category.
(3) For any $\mathbb{K}$-algebra A, let A-bimod be the category of A-bimodules. Settiing $\otimes := \otimes_A$; α the trivial isomorphim; $\mathbf{1} = A$, the regular A-bimodule and l, r the natural isomorphisms $l_B : B \to B \otimes_A A, r_B : B \to A \otimes_A B$ we have that A-bimod is a monoidal category. Note that we have a fully faithful embedding A-bimod $\to \mathrm{End}_{\mathbb{K}}(A\text{-mod})$.
(4) Of course, the category of $\mathbb{K}$-vector spaces is a monoidal category, with $\otimes = \otimes_{\mathbb{K}}$.
(5) For any (pro-)algebraic group G, the category G-rep of finite dimensional representations of G over is a monoidal category, with $\otimes = \otimes_G$.

Let us also recall the notion of a monoidal functor.

Definition 2.1. *Let* $\widetilde{\mathscr{C}} = (\mathscr{C}, \otimes_{\mathscr{C}}, \alpha^{\mathscr{C}}, \mathbf{1}_{\mathscr{C}}, l^{\mathscr{C}}, r^{\mathscr{C}})$, $\widetilde{\mathscr{D}} = (\mathscr{D}, \otimes_{\mathscr{D}}, \alpha^{\mathscr{D}}, \mathbf{1}_{\mathscr{D}}, l^{\mathscr{D}}, r^{\mathscr{D}})$ *be monoidal categories. A* monoidal functor *from* $\widetilde{\mathscr{C}}$ *to* $\widetilde{\mathscr{D}}$ *is a triple* (F, J, J_0) *where:*

(i) $F : \mathscr{C} \to \mathscr{D}$ *is a (covariant) functor.*
(ii) $J : \otimes_{\mathscr{D}} \circ (F \times F) \Rightarrow F \circ \otimes_{\mathscr{C}}$ *is a functor isomorphism.*
(iii) $J_0 : F\mathbf{1}_{\mathscr{C}} \to \mathbf{1}_{\mathscr{D}}$ *is an isomorphism.*

Such that the following diagramas commute for every $X, Y, Z \in \mathrm{ob}(\mathscr{C})$*:*

$$
\begin{tikzcd}
F(X) \otimes_{\mathscr{D}} (F(Y) \otimes_{\mathscr{D}} F(Z)) \arrow[r, "\mathrm{id}_{F(X)} \otimes_{\mathscr{D}} J_{Y,Z}"] \arrow[d, "\alpha^{\mathscr{D}}_{F(X),F(Y),F(Z)}"'] & F(X) \otimes_{\mathscr{D}} F(Y \otimes_{\mathscr{C}} Z) \arrow[r, "J_{X,Y\otimes_{\mathscr{C}} Z}"] & F(X \otimes_{\mathscr{C}} (Y \otimes_{\mathscr{C}} Z)) \arrow[d, "F(\alpha^{\mathscr{C}}_{X,Y,Z})"] \\
(F(X) \otimes_{\mathscr{D}} F(Y)) \otimes_{\mathscr{D}} F(Z) \arrow[r, "J_{X,Y} \otimes_{\mathscr{D}} \mathrm{id}_{F(Z)}"] & F(X \otimes_{\mathscr{C}} Y) \otimes_{\mathscr{D}} F(Z) \arrow[r, "J_{X\otimes_{\mathscr{C}} Y,Z}"] & F((X \otimes_{\mathscr{C}} Y) \otimes_{\mathscr{C}} Z)
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(\mathbf{1}_{\mathscr{C}}) \otimes_{\mathscr{D}} F(X) \arrow[r, "J_{\mathbf{1}_{\mathscr{C}},X}"] \arrow[d, "J_0 \otimes \mathrm{id}_{F(X)}"'] & F(\mathbf{1}_{\mathscr{C}} \otimes_{\mathscr{C}} X) \arrow[d, "F(l^{\mathscr{C}}_X)"] \\
\mathbf{1}_{\mathscr{D}} \otimes_{\mathscr{D}} F(X) \arrow[r, "l^{\mathscr{D}}_{F(X)}"] & F(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \otimes_{\mathscr{D}} F(\mathbf{1}_{\mathscr{C}}) \arrow[r, "J_{X,\mathbf{1}_{\mathscr{C}}}"] \arrow[d, "\mathrm{id}_{F(X)} \otimes_{\mathscr{D}} J_0"'] & F(X \otimes_{\mathscr{C}} \mathbf{1}_{\mathscr{C}}) \arrow[d, "F(r^{\mathscr{C}}_X)"] \\
F(X) \otimes_{\mathscr{D}} \mathbf{1}_{\mathscr{D}} \arrow[r, "r^{\mathscr{D}}_{F(X)}"] & F(X)
\end{tikzcd}
$$

For example, if we consider an algebraic group G and $\mathscr{C} = G$-rep with the monoidal structure defined above, then the forgetful functor $F : \mathscr{C} \to \mathbb{K}$-vect can be given the structure of a monoidal functor, by setting $J_{X,Y} = \mathrm{id}_{X \otimes Y} : X \otimes_{\mathbb{K}} Y \to X \otimes_{\mathbb{K}} Y$ and $J_0 = \mathrm{id}_{\mathbb{K}} : \mathbb{K} \to \mathbb{K}$. Similarly, if H is a quasi-bialgebra, then the forgetful functor H-mod $\to \mathbb{K}$-vect is monoidal.

2.1.2. *Symmetric monoidal categories.* Note that the examples (4) and (5) in the previous subsection satisfy a special symmetry: for $X, Y \in \mathscr{C}$, we have a distinguished isomorphism $X \otimes Y \to Y \otimes X$. In other words, we have a functor isomorphism $\beta : \otimes \to \otimes \circ (12)$, where (12) denotes the functor $\mathscr{C} \times \mathscr{C} \to \mathscr{C} \times \mathscr{C}$ that permutes the factors. Moreover, this functor satisfies $\beta(12) \circ \beta = \mathrm{id}_{\otimes}$. Let us make this property into a definition.

Definition 2.2. *Let $\widetilde{\mathscr{C}} = (\mathscr{C}, \otimes, \alpha, \mathbf{1}, l, r)$ be a monoidal category. A* symmetry *of $\widetilde{\mathscr{C}}$ is an isomorphism of functors $\beta : \otimes \Rightarrow \otimes \circ (12)$ such that $\beta(12) \circ \beta : \otimes \Rightarrow \otimes$ coincides with the identity on $\otimes$ and for every $X, Y, Z \in \mathrm{ob}(\mathscr{C})$ the following diagram commutes:*

$$\begin{tikzcd}
(Z \otimes Y) \otimes X \arrow[r, "\beta_{Z,Y} \otimes \mathrm{id}_X"] \arrow[d, "\alpha^{-1}_{Z,Y,X}"'] & (Y \otimes Z) \otimes X \arrow[r, "\beta_{Y \otimes Z, X}"] & X \otimes (Y \otimes Z) \arrow[d, "\alpha_{X,Y,Z}"] \\
Z \otimes (Y \otimes X) \arrow[r, "\mathrm{id}_Z \otimes \beta_{Y,X}"] & Z \otimes (X \otimes Y) \arrow[r, "\beta_{Z, X \otimes Y}"] & X \otimes (Y \otimes Z)
\end{tikzcd}$$

A symmetric monoidal category *consists of the data $\widetilde{\mathscr{C}} = (\mathscr{C}, \otimes, \alpha, \mathbf{1}, l, r; \beta)$ of a monoidal category $\widetilde{\mathscr{C}}$ and a fixed symmetry β on $\widetilde{\mathscr{C}}$.*

Let us remark that, given any monoidal category $\widetilde{\mathscr{C}} = (\mathscr{C}, \otimes, \alpha, \mathbf{1}, l, r)$ one can form the *opposite monoidal category* $\widetilde{\mathscr{C}}^{\sigma}$ by $\widetilde{\mathscr{C}}^{\sigma} := (\mathscr{C}, \otimes \circ (12), \alpha^{-1} \circ (13), \mathbf{1}, r, l)$. Then, the commutativity of the diagram in Definition 2.2 amounts to saying that $(\mathrm{id}_{\mathscr{C}}, \beta, \mathrm{id}_{\mathbf{1}})$ defines a monoidal functor from $\widetilde{\mathscr{C}}$ to $\widetilde{\mathscr{C}}^{\sigma}$.

We also remark that a symmetric monoidal category is nothing more than a braided monoidal category where the braiding satisfies the extra assumption that it squares to the identity. In particular, given an object $X \in \mathscr{C}$, for every $n \geq 1$ and a choice of parenthesization in $X^{\otimes n}$ we have an action of the Artin braid group B_n on $X^{\otimes n}$. Under the symmetry assumption, it is easy to see that this action factors through the symmetric group S_n.

Now let $\widetilde{\mathscr{C}} = (\mathscr{C}, \otimes_{\mathscr{C}}, \alpha^{\mathscr{C}}, \mathbf{1}_{\mathscr{C}}, l^{\mathscr{C}}, r^{\mathscr{C}}; \beta^{\mathscr{C}})$ and $\widetilde{\mathscr{D}} = (\mathscr{D}, \otimes_{\mathscr{D}}, \mathbf{1}_{\mathscr{D}}, l^{\mathscr{D}}, r^{\mathscr{D}}; \beta^{\mathscr{D}})$ be symmetric monoidal categories. A *symmetric monoidal functor* from $\widetilde{\mathscr{C}}$ to $\widetilde{\mathscr{D}}$ is a monoidal functor (F, J, J_0) compatible with the corresponding symmetries, that is, for every $X, Y \in \mathrm{ob}(C)$ the following diagram commutes:

$$\begin{tikzcd}
F(X) \otimes_{\mathscr{D}} F(Y) \arrow[r, "J_{X,Y}"] \arrow[d, "\beta^{\mathscr{D}}_{F(X),F(Y)}"'] & F(X \otimes_{\mathscr{C}} Y) \arrow[d, "F(\beta^{\mathscr{C}}_{X,Y})"] \\
F(Y) \otimes_{\mathscr{D}} F(X) \arrow[r, "J_{Y,X}"] & F(Y \otimes_{\mathscr{C}} X)
\end{tikzcd}$$

For example, the forgetful functor $F : G\text{-rep} \to \mathbb{K}\text{-vect}$ is a symmetric monoidal functor. This functor satisfies a few more properties that we are going to make into a definition, see Subsection 2.3.

2.2. **Rigidity.** In the category G-rep we have internal homs and duals. We would like to make these concepts. First of all, note that by the usual adjunction formula we have, for $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3 \in G$-rep a canonical isomorphism:

$$\mathrm{Hom}_G(\mathcal{M}_1 \otimes_G \mathcal{M}_2, \mathcal{M}_3) \cong \mathrm{Hom}_G(\mathcal{M}_1, \mathrm{Hom}_{\mathbb{K}}(\mathcal{M}_2, \mathcal{M}_3))$$

In other words, the functor $\mathrm{Hom}_G(\bullet \otimes_G \mathcal{M}, \mathcal{N}) : G\text{-rep}^{\mathrm{opp}} \to \mathfrak{Sets}$ is represented by the object $\mathrm{Hom}_{\mathbb{K}}(\mathcal{M}, \mathcal{N})$. This is a notion that makes sense in every tensor category, so we make it into a definition.

Definition 2.3. *Let $\widetilde{\mathscr{C}}$ be a monoidal category, and let $X, Y \in \mathrm{ob}(\mathscr{C})$. If the functor*

$$\mathrm{Hom}_{\mathscr{C}}(\bullet \otimes X, Y) : \mathscr{C}^{\mathrm{opp}} \to \mathfrak{Sets}$$

is representable, then we denote the representing object by $\underline{\mathrm{Hom}}(X, Y)$ and call it the internal hom *of X and Y.*

As we have seen, for the category G-rep internal homs always exist, and $\underline{\mathrm{Hom}}(\mathcal{M}, \mathcal{N}) = \mathrm{Hom}_{\mathbb{K}}(\mathcal{M}, \mathcal{N})$. Note that we have a natural isomorphism $\mathrm{Hom}_{\mathscr{C}}(\underline{\mathrm{Hom}}(X, Y) \otimes X, Y) \to \mathrm{Hom}_{\mathscr{C}}(\underline{\mathrm{Hom}}(X, Y), \underline{\mathrm{Hom}}(X, Y))$. The *evaluation map* $\mathrm{ev}_{X,Y} : \underline{\mathrm{Hom}}(X, Y) \otimes X \to Y$ is the map corresponding to $\mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}$. Thus, for every morphism $f : T \otimes X \to Y$, there exists a unique morphism $g : T \to \underline{\mathrm{Hom}}(X, Y)$ such that the following diagram commutes:

$$\begin{tikzcd} T \otimes X \arrow[d, "g\otimes \mathrm{id}_X"'] \arrow[drr, "f"] & & \\ \underline{\mathrm{Hom}}(X,Y) \otimes X \arrow[rr, "\mathrm{ev}_{X,Y}"] & & Y \end{tikzcd}$$

Now assume that $\underline{\mathrm{Hom}}(X, Y)$ exists for any objects $X, Y \in \mathrm{ob}(\mathscr{C})$. Let $X, Y, Z \in \mathrm{ob}(\mathscr{C})$. Then, for every $T \in \mathrm{ob}(\mathscr{C})$ we have a sequence of natural bijections:

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(T, \underline{\mathrm{Hom}}(Z, \underline{\mathrm{Hom}}(X, Y))) &\cong \mathrm{Hom}_{\mathscr{C}}(T \otimes Z, \underline{\mathrm{Hom}}(X, Y)) \\ &\cong \mathrm{Hom}_{\mathscr{C}}((T \otimes Z) \otimes X, Y) \\ &\cong \mathrm{Hom}_{\mathscr{C}}(T \otimes (Z \otimes X), Y) \\ &\cong \mathrm{Hom}_{\mathscr{C}}(T, \underline{\mathrm{Hom}}(Z \otimes X, Y)) \end{aligned}$$

So that we have a natural isomorphism $\underline{\mathrm{Hom}}(Z \otimes X, Y) \cong \underline{\mathrm{Hom}}(Z, \underline{\mathrm{Hom}}(X, Y))$. Now we define the dual of an object X.

Definition 2.4. *Let $\widetilde{\mathscr{C}}$ be a monoidal category and $X \in \mathrm{ob}(\mathscr{C})$. The* dual *of X, denoted by $X^\vee$, is defined to be $\underline{\mathrm{Hom}}(X, \mathbf{1})$, in case the latter object exists.*

If the dual object $X^\vee$ exists, we denote the evaluation map simply by $\mathrm{ev}_X : X^\vee \otimes X \to \mathbf{1}$. There is also the notion of a *coevaluation* map, $\mathrm{coev}_X : \mathbf{1} \to \underline{\mathrm{Hom}}(X, X)$. Indeed, $\mathrm{Hom}_{\mathscr{C}}(\mathbf{1}, \underline{\mathrm{Hom}}(X, X)) = \mathrm{Hom}_{\mathscr{C}}(X, X)$, and the coevaluation map is nothing more than the map corresponding to id_X.

Assume for a moment now that $\widetilde{\mathscr{C}}$ is symmetric (or, more generally, braided) with symmetry given by β. We have a map $\mathrm{ev}_X \circ \beta_{X,X^\vee} : X \otimes X^\vee \to \mathbf{1}$. Since $\mathrm{Hom}_{\mathscr{C}}(X \otimes X^\vee, \mathbf{1}) = \mathrm{Hom}_{\mathscr{C}}(X, (X^\vee)^\vee)$, the map $\mathrm{ev}_X \circ \beta_{X,X^\vee}$ corresponds to a unique map $\iota_X \in \mathrm{Hom}_{\mathscr{C}}(X, (X^\vee)^\vee)$. We say that X is *reflexive* if ι_X is an isomorphism. For example, if G is an algebraic group then every object in G-rep is reflexive.

We remark that there is also a notion of dual maps. Assume both X, Y have duals, and let $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. Then, by the definition of the evaluation map ev_X there exists a unique map $f^\vee : Y^\vee \to X^\vee$ that makes the following diagram commutative:

$$\begin{tikzcd} Y^\vee \otimes X \arrow[r, "f^\vee \otimes \mathrm{id}_X"] \arrow[d, "\mathrm{id}_{Y^\vee} \otimes f"'] & X^\vee \otimes X \arrow[d, "\mathrm{ev}_X"] \\ Y^\vee \otimes Y \arrow[r, "\mathrm{ev}_Y"] & \mathbf{1} \end{tikzcd}$$

We remark that, if X, Y are reflexive, then by definition we get $(f^\vee)^\vee = f$.

Under the assumption that $\widetilde{\mathscr{C}}$ is symmetric, for every family of pairs $\{(X_i, Y_i)\}_{i=1}^n$ we have a natural map:

$$\left(\bigotimes_{i=1}^n \underline{\mathrm{Hom}}(X_i, Y_i)\right) \otimes \left(\bigotimes_{i=1}^n X_i\right) \stackrel{\cong}{\longrightarrow} \bigotimes_{i=1}^n (\underline{\mathrm{Hom}}(X_i, Y_i) \otimes X_i) \xrightarrow{\otimes_{i=1}^n \mathrm{ev}_{X_i,Y_i}} \bigotimes_{i=1}^n Y_i$$

that corresponds to a map

$$\bigotimes_{i=1}^n \underline{\mathrm{Hom}}(X_i, Y_i) \longrightarrow \bigotimes_{i=1}^n \underline{\mathrm{Hom}}\left(\bigotimes_{i=1}^n X_i, \bigotimes_{i=1}^n Y_i\right) \tag{4}$$

Definition 2.5. *Let $\widetilde{\mathscr{C}}$ be a symmetric monoidal category. Then, $\widetilde{\mathscr{C}}$ is said to be* rigid *if the following conditions are satisfied.*

(1) *The internal hom $\underline{\mathrm{Hom}}(X, Y)$ exists for any two objects $X, Y \in \mathrm{ob}(\mathscr{C})$.*
(2) *Every object $X \in \mathrm{ob}(\mathscr{C})$ is reflexive.*
(3) *For any family of pairs $\{(X_i, Y_i)\}_{i=1}^n$ of objects of $\mathscr{C}$, the map (4) is an isomorphism.*

For example, the category of finite-dimensional representations of an algebraic group G is rigid. On the other hand, if A is a $\mathbb{K}$-algebra then the category A-mod is not necessarily rigid. Indeed, here we have that $\underline{\mathrm{Hom}}(X,Y) = \mathrm{Hom}_A(X,Y)$, so there could be nonzero objects X with $X^\vee = \mathrm{Hom}_A(X,A) = 0$. The next proposition follows easily from the definition of a rigid category.

Proposition 2.6. *Let $\widetilde{\mathscr{C}}$ be a rigid, symmetric monoidal category. Then, we have isomorphisms*

- $X^\vee \otimes Y \to \underline{\mathrm{Hom}}(X,Y)$, *and*
- $\bigotimes_{i=1}^n X_i^\vee \to (\bigotimes_{i=1}^n X_i)^\vee$.

for every objects $X_1, \dots, X_n, X, Y \in \mathrm{ob}(\mathscr{C})$.

We remark that, in particular, we may interpret the coevaluation map as $\mathrm{coev}_X : \mathbf{1} \to X^\vee \otimes X$.

Proposition 2.7. *Let $\widetilde{\mathscr{C}}$, $\widetilde{\mathscr{D}}$ be rigid, symmetric monoidal categories, and let $F, G : \widetilde{\mathscr{C}} \to \widetilde{\mathscr{D}}$ be monoidal functors. Then, every morphism $\Phi : F \Rightarrow G$ is an isomorphism of functors.*

Proof. An inverse for Φ can be constructed using the following commutative diagram:

$$\begin{tikzcd} F(X^\vee) \arrow[r, "\Phi_{X^\vee}"] \arrow[d, "\cong"] & G(X^\vee) \arrow[d, "\cong"'] \\ F(X)^\vee \arrow[r, "(\Phi_X^{-1})^\vee"] & G(X)^\vee \end{tikzcd}$$

$\square$

Proposition 2.8. *Let $\widetilde{\mathscr{C}}$ be a rigid, symmetric, monoidal category, and let $X \in \mathrm{ob}(\mathscr{C})$. Then, for every objects $Y, Z \in \mathrm{ob}(\mathscr{C})$ we have natural isomorphisms:*

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(Y \otimes X, Z) &\overset{\cong}{\to} \mathrm{Hom}_{\mathscr{C}}(Y, Z \otimes X^\vee) \\ \mathrm{Hom}_{\mathscr{C}}(X \otimes Y, Z) &\overset{\cong}{\to} \mathrm{Hom}_{\mathscr{C}}(Y, X^\vee \otimes Z) \end{aligned} \tag{5}$$

Proof. Let us give a formula for the first isomorphism. For $f \in \mathrm{Hom}_{\mathscr{C}}(Y \otimes X, Z)$, its image is given by the following diagram:

$$Y \xrightarrow{\mathrm{id}_Y \otimes \mathrm{coev}_{X^\vee}} Y \otimes X \otimes X^\vee \xrightarrow{f \otimes \mathrm{id}_{X^\vee}} Z \otimes X^\vee$$

Its inverse is as follows. For $g \in \mathrm{Hom}_{\mathscr{C}}(Y, Z \otimes X^\vee)$, its image is given by the following diagram:

$$Y \otimes X \xrightarrow{g \otimes \mathrm{id}_X} Z \otimes X^\vee \otimes X \xrightarrow{\mathrm{id}_Z \otimes \mathrm{ev}_X} Z$$

Finally, thanks to symmetry the second isomorphism follows from the first one. $\square$

Note that Equations (5) tell us that the functor $\bullet \otimes X$ is left adjoint to $\bullet \otimes X^\vee$. Since X is reflexive, these functors are biadjoint. Similarly, $X \otimes \bullet$ and $X^\vee \otimes \bullet$ are biadjoint functors. This has the following immediate corollary.

Proposition 2.9. *Let $\widetilde{\mathscr{C}}$ be an abelian, rigid, symmetric monoidal category. Then, $\otimes$ commutes with inverse and direct limits in each variable. In particular, it is exact on both variables.*

2.3. **Fiber functors.** We have axiomatized many properties of the category G-rep. The key property in Theorem 1.12, however, is the existence of a "nice" functor $F : G\text{-rep} \to \mathbb{K}\text{-vect}$. Let us be more explicit about what we mean by "nice".

Definition 2.10. *Let $\widetilde{\mathscr{C}} = (\mathscr{C}, \otimes, \alpha, \mathbf{1}, l, r; \beta)$ be a symmetric, abelian monoidal category. A* fiber functor *is an exact and faithful monoidal functor $(F, J, J_0) : \widetilde{\mathscr{C}} \to \mathbb{K}\text{-vect}$. A* Tannakian category *is an abelian, rigid, symmetric monoidal category admitting a fiber functor.*

Of course, the category G-rep is an example of a Tannakian category. Surprisingly enough, the main theorem of this section tells us that this is, basically, the only example of a Tannakian category.

2.4. **Tannakian categories vs. Algebraic groups.** Let us remark that if $\mathscr{C}$ is a monoidal $\mathbb{K}$-linear category and $F : \mathscr{C} \to \mathbb{K}\text{-vect}$ is a tensor functor, then for every algebra A we have an induced tensor functor $F_A : \mathscr{C} \to A\text{-mod}$, $F_A(X) := F(X) \otimes_{\mathbb{K}} A$. In particular, similarly to Subsection 1.4 we obtain a $\mathbb{K}$-group functor $\mathrm{Aut}^{\otimes}(F) : \mathbb{K}\text{-alg} \to \mathfrak{Groups}$. The main objective of this subsection is to prove the following remarkable theorem.

Theorem 2.11. *Let $\widetilde{\mathscr{C}}$ be a rigid, abelian, symmetric monoidal $\mathbb{K}$-linear category and let $F : \widetilde{\mathscr{C}} \to \mathbb{K}\text{-vect}$ be a fiber functor. Then:*

(a) *The functor $\mathrm{Aut}^{\otimes}(F)$ is an algebraic group, say G.*
(b) *The functor $\widetilde{\mathscr{C}} \to G\text{-rep}$ induced by F is an equivalence of categories.*

Let us sketch the main ideas of the proof of Theorem 2.11. The first idea is to use an argument similar to that of the proof of Theorem 1.12, that is, restrict to a subcategory of $\widetilde{\mathscr{C}}$ "tensor subgenerated by a single object" and prove a version of Theorem 2.11 for this category. In fact, we will prove a weaker version of this: we will prove that the category $\widetilde{\mathscr{C}}_X$ is equivalent to the category of comodules over a coalgebra, we will do this in Subsection 2.4.2 after giving some linear algebraic preliminaries in Subsection 2.4.1. Using an inverse limit argument, this will imply that the entire category $\widetilde{\mathscr{C}}$ is equivalent to the category of comodules over a coalgebra, and we need to check that this is actually a commutative Hopf algebra, this is done in Subsection 2.4.3.

2.4.1. *Preliminaries.* Before proceeding to the proof of Theorem 2.11, we give some linear algebraic preliminaries. For the rest of this section We keep the notation of Theorem 2.11.

Proposition 2.12. *There exists a functor*

$$\bullet \boxtimes \bullet : \mathbb{K}\text{-vect} \times \mathscr{C} \to \mathscr{C}$$

satisfying the following properties.

(1) $\mathrm{Hom}_{\mathscr{C}}(T, V \boxtimes X) \cong V \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathscr{C}}(T, X)$ *and* $\mathrm{Hom}_{\mathscr{C}}(V \boxtimes X, T) \cong V \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathscr{C}}(X, T)$ *(functorially in T).*
(2) *For any $\mathbb{K}$-linear functor $F' : \mathscr{C} \to \mathbb{K}\text{-vect}$, $F'(V \boxtimes X) \cong V \otimes_{\mathbb{K}} F'(X)$.*

For $X, T \in \mathrm{ob}(\mathscr{C})$ and $V \in \mathbb{K}\text{-vect}$.

Proof. We give a a construction of the functor $\boxtimes$. First of all, we pick a skeleton of the category $\mathbb{K}$-vect: this is given by vector spaces of the form $\mathbb{K}^n$ for $n \in \mathbb{Z}_{\geq 0}$. Let us call this skeleton $\mathbb{K}\text{-vect}^s$. For each finite dimensional vector space V, choose an isomorphism $\delta_V : \mathbb{K}^{\dim V} \to V$. Since $\mathbb{K}\text{-vect}^s$ is an skeleton of $\mathbb{K}$-vect, there exists a unique functor $\Gamma : \mathbb{K}\text{-vect} \to \mathbb{K}\text{-vect}^s$ such that Γ is an equivalence of categories, with quasi-inverse given by the inclusion $\iota : \mathbb{K}\text{-vect}^s \to \mathbb{K}\text{-vect}$ and δ provides a natural isomorphism $\delta : \gamma \circ \iota \to \mathrm{id}_{\mathbb{K}\text{-vect}}$.

The construction of the functor $\boxtimes$ is now easy. First of all, define $\mathbb{K}^n \boxtimes X := X^{\oplus n}$. For a $V \in \mathbb{K}\text{-vect}$, define $V \boxtimes X = \gamma(V) \boxtimes X$. It is straightforward to see that $\boxtimes$ satisfies properties (1) and (2) above. ☐

For a vector space V and $X \in \mathscr{C}$, we define

$$\underline{\mathrm{Hom}}(V, X) := V^{\vee} \boxtimes X,$$

we remark that, although we are using the same notation for the internal hom here, in this construction the first argument is always a vector space V, while in the internal tensor product the first argument is an object of $\mathscr{C}$, so there is no risk of confusion. Now, if $W \subseteq V$ is a subspace and $Y \subseteq X$ is a subobject, we would like to define a subspace of $\underline{\mathrm{Hom}}(V, X)$ consisting of "all maps mapping W into Y". Of course, this does not make sense as stated because $\underline{\mathrm{Hom}}(V, X)$ is not technically a space of maps. If it were a space of maps, however, a way to rephrase "all maps mapping W into Y" is as "all maps such that the composition $W \to X \to X/Y$ is 0". Thus, we define the *transporter*:

$$(Y : W) := \ker(\underline{\mathrm{Hom}}(V, X) \to \underline{\mathrm{Hom}}(W, X/Y))$$

note that we have that $F(\underline{\mathrm{Hom}}(V, X)) \cong \mathrm{Hom}_{\mathbb{K}}(V, F(X))$ and, since F is exact, $F(Y : W) = (F(Y) : W) = \{f \in \mathrm{Hom}_{\mathbb{K}}(V, F(X)) : f(W) \subseteq F(Y)\}$.

Now, in the situation of Theorem 2.11, we would like that the restriction of F to $\widetilde{\mathscr{C}}_X$ identifies $\widetilde{\mathscr{C}}_X$ with a category of comodules over a certain coalgebra, say C. If we assume that C is finite dimensional, then the category of comodules over C is equivalent to the category of modules over $C^\vee$. So we would like to see that $\widetilde{\mathscr{C}}_X$ is equivalent to the category of modules over a certain algebra R. A first good candidate is $\mathrm{Hom}_{\mathbb{K}}(F(X), F(X))$. This algebra, however, is too big and will fail to preserve the subojects: if Y is a subobject of X, $F(Y)$ is not necessarily a $\mathrm{Hom}_{\mathbb{K}}(F(X), F(X))$-submodule of $F(X)$. The next result constructs a good candidate for our algebra R.

Lemma 2.13. *Under the assumptions of Theorem 2.11, for any object $X \in \mathrm{ob}(\mathscr{C})$ the following two subojects of $\underline{\mathrm{Hom}}(F(X), X)$ are equal.*

1. *The largest subobject $P \subseteq \underline{\mathrm{Hom}}(F(X), X)$ whose image in $\underline{\mathrm{Hom}}(F(X)^{\oplus n}, X^{\oplus n})$ (under the diagonal embedding) is contained in $(Y : F(Y))$ for any $Y \subseteq X^{\oplus n}$ and any $n \geq 0$.*
2. *The smallest subobject $P' \subseteq \underline{\mathrm{Hom}}(F(X), X)$ such that $F(P') \subseteq F(\underline{\mathrm{Hom}}(F(X), X)) = \mathrm{Hom}_{\mathbb{K}}(F(X), F(X))$ contains $\mathrm{id}_{F(X)}$.*

Proof. First, we remark that the existence of the functor F implies that every object of $\mathscr{C}$ is artinian and noetherian. In particular, both P and P' are well-defined subobjects of $\underline{\mathrm{Hom}}(F(X), X)$. Now, by definition:

$$P = \bigcap_{n \geq 0} \bigcap_{Y \subseteq X^{\oplus n}} (\underline{\mathrm{Hom}}(F(X), X) \cap (Y : F(Y))) \Rightarrow F(P) = \bigcap_{n \geq 0} \bigcap_{Y \subseteq X^{\oplus n}} (\mathrm{End}_{\mathbb{K}}(F(X)) \cap (F(Y) : F(Y)))$$

So $F(P)$ is the largest subring of $\mathrm{End}_{\mathbb{K}}(F(X))$ stabilizing all $F(Y)$, $Y \subseteq X^{\oplus n}$. So $\mathrm{id}_X \in F(P)$ and $P' \subseteq P$.

Now, consider the space $\underline{\mathrm{Hom}}(F(X), X)$. If $Y \subseteq \underline{\mathrm{Hom}}(F(X), X)$ is a subobject then, by the definition of P, left multiplication by $F(P) \subseteq \mathrm{End}_{\mathbb{K}}(F(X))$ stabilizes $F(Y) \subseteq \mathrm{End}_{\mathbb{K}}(F(X))$. Since $1_{F(X)} \in F(P')$, we see that $F(P) \subseteq F(P')$, so $P \subseteq P'$. □

2.4.2. *Constructing a coalgebra.* Consider $X \in \mathrm{ob}(\mathscr{C})$. Let $P_X \subseteq \underline{\mathrm{Hom}}(F(X), X)$ be the object defined by Lemma 2.13. If $\langle X \rangle \subseteq \mathscr{C}$ denotes the subcategory of objects subgenerated by X (= subobjects of quotients of $X^{\oplus n}$) then, by definition, the functor $F|_{\langle X \rangle} : \langle X \rangle \to \mathbb{K}$-vect factors through $F(P_X)$-mod. We remark that $\langle X \rangle$ is in general *not* closed under tensor products. Let us denote $F(P_X) := A_X$.

Proposition 2.14. *For any $Y \in \mathrm{ob}(\langle X \rangle)$, there is a natural action of A_X on $F(Y)$. Moreover, $F|_{\langle X \rangle} : \langle X \rangle \to A_X$-mod is an equivalence of categories sending $F|_{\langle X \rangle}$ to the forgetful functor, and $A_X = \mathrm{End}(F|_{\langle X \rangle})$.*

Proof. The first assertion is clear. Note that we have an action of $A_X \subseteq \mathrm{End}(F(X)^\vee)$ on $\underline{\mathrm{Hom}}(F(X), X) = F(X)^\vee \boxtimes X$ and it is clear that this action stabilizes P_X. Now, if M is a right A-module then we get two maps $((M \otimes A_X) \boxtimes P_X) \to M \boxtimes P_X$, one by considering the action of A_X on M and the other by considering the action on P_X. We define $M \otimes_{A_X} P_X$ to be the equalizer of these maps. Note that, by definition, it is an object of $\langle X \rangle$.Then we have:

$$F(M \otimes_{A_X} P_X) = M \otimes_{A_X} F(P_X) = M$$

so F is essentially surjective. Now, if $f : M \to N$ is an A_X-module map, then we may define a map $M \otimes_{A_X} P_X \to N \otimes_{A_X} P_X$ that sees that F is full. Finally, F is faithful by hypothesis. So F is a category equivalence. The last assertion of the proposition is easy. □

Now let $C_X := A_X^\vee$, so that $\langle X \rangle$ is equivalent to the category of C-comodules. Thus, we get:

Proposition 2.15. *Let $H := \varinjlim \mathrm{End}(F|_{\langle X \rangle})^\vee$. Then, F factors through the category of H-comodules and, moreover, it is an equivalence between $\mathscr{C}$ and the category of H-comodules carrying F into the forgetful functor.*

Note that we really have not used that $\widetilde{\mathscr{C}}$ is a symmetric category. We will use this to define a commutative Hopf algebra structure on H.

2.4.3. *H is a Hopf algebra.* Now let B be *any* $\mathbb{K}$-coalgebra, and consider the forgetful functor $\omega : B$-comod $\to$ $\mathbb{K}$-vect, where B-comod is the category of finite dimensional B-comodules. Note that $B = \varinjlim_X \mathrm{End}(\omega|_{\langle X\rangle})^\vee$. This, and the fact that for a finite dimensional algebra A, $A \cong \mathrm{End}(\omega_A)$, implies that any functor B-comod $\to B'$-comod that carries the forgetful functor to itself arises from a unique coalgebra homomorphism $B \to B'$.

Now assume that we have a coalgebra homomorphism $f : B \otimes_{\mathbb{K}} B \to B$. Clearly, this defines a functor $\phi^f : B$-comod $\times B$-comod, $(X, Y) \mapsto X \otimes_{\mathbb{K}} Y$ with comodule structure defined by f. Even more is true.

Lemma 2.16. *The map $f \mapsto \phi^f$ defines a one-to-one correspondence between the set of coalgebra homomorphisms $B \otimes_{\mathbb{K}} B \to B$ and the set of functor B*-comod $\times B$-comod $\to B$-comod *such that $(X, Y) \mapsto X \otimes_{\mathbb{K}} Y$ as vector spaces. The product induced by f is associative (resp. commutative) if and only if the natural associativity (resp. commutativity) constraint on* $\mathbb{K}$-vect *induces a similar constrant on B*-comod. *The product induced by f has a unit if and only if B*-comod *has a unit object with underlying vector space $\mathbb{K}$.*

Returning to the setting of the proof of Theorem 2.11, the previous Lemma immediately shows that H is a commutative algebra with identity. Note that $G = \mathrm{Spec}(H)$ is a *monoid* scheme, that is, it is a functor from $\mathbb{K}$-alg to the category of monoids. Similarly to Subsection 1.4. we can show that $G \cong \mathrm{End}^{\otimes}(F)$. Since both $\widetilde{\mathscr{C}}$ and $\mathbb{K}$-vect are rigid, Proposition 2.7 shows that $\mathrm{End}^{\otimes}(F) = \mathrm{Aut}^{\otimes}(F)$. So G is actually a group scheme and H is a Hopf algebra. We are done with the proof of Theorem 2.11.

2.5. **Examples.** Let us give a few examples of Tannakian categories and their respective algebraic groups.

2.5.1. *Graded vector spaces.* Let $\mathscr{C}$ be the category of graded vector spaces: its objects are families $(V^n)_{n\in\mathbb{Z}}$ of $\mathbb{K}$-vector spaces with finite dimensional direct sum $V := \bigoplus_{n\in\mathbb{Z}} V^n$. The unit $\mathbf{1}$ is the graded vector space with $V^n = \mathbb{K}^{\delta_{n,0}}$. We have a forgetful functor $F : (V_n) \to V$. The internal hom is $\underline{\mathrm{Hom}}((V^n), W^m) = (\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}(V^i, W^{i+n}))_n$, and the tensor product is $(V^n) \otimes (W^m) = (\bigoplus_i V^i \otimes_{\mathbb{K}} W^{n-i})_n$. This is equivalent to the category of representations of the algebraic group $\mathbb{G}_m$. Namely, (V^n) corresponds to the representation V of $\mathbb{G}_m$ for which G_m acts on V^n via the character $\lambda \mapsto \lambda^n$.

2.5.2. *Hodge structures.* Now let $\mathbb{K} = \mathbb{R}$. A *real Hodge structure* is a finite-dimensional $\mathbb{K}$-vector space V with a decomposition of real vector spaces

$$V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q} V^{p,q}$$

such that $V^{p,q}$ and $V^{q,p}$ are complex conjugates subspaces of $V \otimes_{\mathbb{R}} \mathbb{C}$. The category $\mathscr{C}$ now is the category of real Hodge structures, with $F : (V, V^{p,q}) \to V$ being the fiber functor. This category is equivalent to the category of representations of the real algebraic group $\mathbb{S}$ that is the restriction of scalars of $\mathbb{G}_m$ from $\mathbb{C}$ to $\mathbb{R}$, namely, $\mathbb{R}[\mathbb{S}] = \mathbb{C}[x^{\pm 1}]$ considered as an $\mathbb{R}$-algebra. The real Hodge structure $(V, (V^{p,q}))$ corresponds to the representation of $\mathbb{S}$ on V such that $\lambda \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ acts on $V^{p,q}$ by $\lambda^{-p}\overline{\lambda}^{-q}$.

3. Deligne's Theorem

We have seen that if $\widetilde{\mathscr{C}}$ is an abelian, rigid, symmetric monoidal category admitting a fiber functor $F : \widetilde{\mathscr{C}} \to \mathbb{K}$-vect, then $\widetilde{\mathscr{C}}$ is equivalent to the category of representations of an algebraic group G. However, a fiber functor is not always easy to construct, so we would like to have more intrinsic conditions on the category $\widetilde{\mathscr{C}}$ that ensure the existence of such a functor. This is what we do in this section. These conditions are based on linear algebraic constructions that are valid in *any* abelian, rigid, symmetric monoidal category. We will study linear algebra in this more general setting in Subsection 3.1. In particular, for any object $X \in \mathrm{ob}(\mathscr{C})$ we have a notion of the *dimension* of X. As we will see, dimensions are preserved by monoidal functors. So a necessary condition for the existence of a fiber functor is that the dimension of any nonzero object is a positive integer. Surprisingly enough, this condition turns out to be also sufficient. We will sketch a proof of this in Subsection 3.2.

3.1. **Linear algebra on monoidal categories.** From now and until the end of these notes, we fix an abelian, rigid, symmetric, monoidal category $\widetilde{\mathscr{C}} = (\mathscr{C}, \otimes, \alpha, \mathbf{1}, l, r; \beta)$. For simplicity, we will also assume that $\mathscr{C}$ is Karoubian, that is, idempotents split in $\mathscr{C}$, that $\mathbb{K}$ is of characteristic zero and, moreover, that $\mathrm{End}_{\mathscr{C}}(\mathbf{1}) = \mathbb{K}$.

3.1.1. *Traces and dimension.* Recall that if $X \in \mathrm{ob}(\mathscr{C})$ then we denote its dual object by $X^\vee := \underline{\mathrm{Hom}}(X, \mathbf{1})$. Since $\mathscr{C}$ is rigid, we have an isomorphism $\varphi_X : \underline{\mathrm{Hom}}(X, X) \to X^\vee \otimes X$. On the other hand, we have the evaluation map $\mathrm{ev}_X : X^\vee \otimes X \to \mathbf{1}$. We call $\mathrm{tr}_X := \mathrm{ev}_X \circ \varphi_X : \underline{\mathrm{Hom}}(X, X) \to \mathbf{1}$ the *trace morphism* of X.

We define the *dimension* of X to be $\dim X := \mathrm{tr}_X \circ \mathrm{coev}_X : \mathbf{1} \to \mathbf{1}$. By our assumptions, $\dim X$ is an element of the field $\mathbb{K}$. We remark that the dimension is additive, $\dim(X) = \dim(Y) + \dim(X/Y)$, and multiplicative, $\dim(X \otimes Y) = \dim(X)\dim(Y)$, this follows easily from left and right exactness of $\otimes$. We also remark that, by uniqueness of dual objects and canonicity of the maps ev_X, φ_X and coev_X, if $F : \widetilde{\mathscr{C}} \to \widetilde{\mathscr{D}}$ is a tensor functor between abelian, rigid, symmetric monoidal categories, then $\dim(X) = \dim(F(X))$ for any $X \in \mathrm{ob}(\mathscr{C})$. In particular, *if there exists a monoidal functor $F : \widetilde{\mathscr{C}} \to \mathbb{K}\text{-vect}$, then the dimension of any nonzero object of $\mathscr{C}$ is a positive integer.*

3.1.2. *Symmetric and exterior powers.* Recall that a choice of symmetry β determines an action of the symmetric group S_n on $X^{\otimes n}$ for any object $X \in \mathrm{ob}(\mathscr{C})$. We have the symmetrization map:

$$s := \frac{1}{n!}\sum_{\sigma \in S_n} \sigma : X^{\otimes n} \to X^{\otimes n}$$

and the antisymmetrization map:

$$a := \frac{1}{n!}\sum_{\sigma \in S_n} \mathrm{sign}(\sigma)\sigma : X^{\otimes n} \to X^{\otimes n}$$

Note that both s and a are idempotents in $\mathrm{End}_{\mathscr{C}}(X^{\otimes n})$. Since we are assuming our category is Karoubian, they determine direct summands of $X^{\otimes n}$. We define the *symmetric power*, $S^n(X)$, to be the direct summand determined by s; and the *exterior power*, $\Lambda^n(X)$, to be the direct summand determined by a. The usual dimension formulas hold:

$$\dim S^n(X) = \frac{\prod_{i=0}^{n-1}(\dim(X) + i)}{n!}, \qquad \dim \Lambda^n(X) = \frac{\prod_{i=0}^{n-1}(\dim(X) - i)}{n!}$$

in particular, if $\dim(X) = d$, a positive integer, then $\dim(\Lambda^n(X)) = 0$ for $n > d$.

3.1.3. *Algebras and modules.* Recall that an algebra object of the category $\widetilde{\mathscr{C}}$ is a functor from the prop **Alg** to $\widetilde{\mathscr{C}}$. In other words, it consists of an object A and maps $\mu : A \otimes A \to A$, $\eta : \mathbf{1} \to A$ satisfying the usual properties. If A is an algebra object, then a left A-module is an object $M \in \mathrm{ob}(\mathscr{C})$ together with a map $m : A \otimes M \to M$ such that the following diagrams commute:

$$\begin{tikzcd}
M \arrow[r, "l_M"] \arrow[d, "\mathrm{id}_M"'] & \mathbf{1} \otimes M \arrow[d, "\eta\otimes\mathrm{id}_M"] \\
M & A \otimes M \arrow[l, "m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes (A \otimes M) \arrow[rr, "\mathrm{id}_A \otimes m"] \arrow[d, "\alpha_{A,A,M}"'] & & A \otimes M \arrow[d, "m"] \\
(A \otimes A) \otimes M \arrow[r, "\mu\otimes\mathrm{id}_M"] & A \otimes M \arrow[r, "m"] & M
\end{tikzcd}$$

For example, if $M \in \mathrm{ob}(\mathscr{C})$ is any object, then $A \otimes M$ acquires a natural A-module structure.

We will denote by $\widetilde{\mathscr{C}}$-alg the category of algebra objects of the category $\widetilde{\mathscr{C}}$, the notion of a morphism in $\widetilde{\mathscr{C}}$-alg is clear. A $\widetilde{\mathscr{C}}$-algebra A is *commutative* if $\mu \circ \beta_{A,A} = \mu$. We denote by $\widetilde{\mathscr{C}}\text{-alg}^\beta$ the category of commutative $\mathscr{C}$-algebras.

We have a forgetful functor $F : \widetilde{\mathscr{C}}\text{-alg} \to \widetilde{\mathscr{C}}$ that forgets the algebra structure. This functor admits a left adjoint, $T : \widetilde{\mathscr{C}} \to \widetilde{\mathscr{C}}\text{-alg}$, for $X \in \mathrm{ob}(\mathscr{C})$, $T(X) = \bigoplus_{n \geq 0} X^{\otimes n}$, and the multiplication $X^{\otimes n} \otimes X^{\otimes m} \to X^{\otimes (n+m)}$ is given by the associativity constraint. The forgetful functor $F : \widetilde{\mathscr{C}}\text{-alg}^\beta \to \widetilde{\mathscr{C}}$ also admits a right adjoint $S : \widetilde{\mathscr{C}} \to \widetilde{\mathscr{C}}\text{-alg}^\beta$, $S(X) = \bigoplus_{n\geq 0} S^n(X)$. The following technical lemmas are going to be important in the proof of our main theorem.

Lemma 3.1. *Let $X \in \mathrm{ob}(\mathscr{C})$. Assume that $\dim(X) \notin \mathbb{Z}_{\leq 0} \subseteq \mathbb{K}$ (recall we assume that $\mathbb{K}$ is of characteristic zero.) Then, there exists a commutative $\widetilde{\mathscr{C}}$-algebra B (more precisely, B belongs to the ind-completion of $\widetilde{\mathscr{C}}$) such that the B-module $B \otimes X$ is isomorphic to $B \oplus N$ as B-modules. Moreover, $B \otimes Y \neq 0$ for any object Y of $\widetilde{\mathscr{C}}$.*

Proof. We want to show that there exists an algebra B such that $\{(u,v) \in \mathrm{Hom}_B(B, B\otimes X) \times \mathrm{Hom}_B(B\otimes X, B) : v\circ u = \mathrm{id}_B\} \neq \emptyset$. We proceed in several steps.

Step 1. Consider the functor $G : \widetilde{\mathscr{C}}\text{-alg}^\beta \to \mathfrak{Sets}$, $A \mapsto \mathrm{Hom}_A(A, A\otimes X) \times \mathrm{Hom}_A(A\otimes X, A)$. By the A-module structure on $A\otimes X$, G is isomorphic to $G' : \widetilde{\mathscr{C}}\text{-alg}^\beta \to \mathfrak{Sets}$, $A \mapsto \mathrm{Hom}_{\mathscr{C}}(X, A) \times \mathrm{Hom}_{\mathscr{C}}(X^\vee, A) = \mathrm{Hom}_{\mathscr{C}}(X\oplus X^\vee, A)$. By adjointness, this functor is isomorphic to $G'' : \widetilde{\mathscr{C}}\text{-alg}^\beta \to \mathfrak{Sets}$, $A \mapsto \mathrm{Hom}_{\widetilde{\mathscr{C}}\text{-alg}^\beta}(S(X\oplus X^\vee), A)$.

Step 2. We have a natural morphism of functors $c : G \to H$, where $H(A) = \mathrm{Hom}_A(A, A) = \mathrm{Hom}_{\mathscr{C}}(\mathbf{1}, A) = \mathrm{Hom}_{\widetilde{\mathscr{C}}\text{-alg}^\beta}(S(\mathbf{1}), A)$ which is given by composition. So we get a morphism of functors $c : G'' \to H$, which thanks to the Yoneda Lemma is associated to an algebra morphism $c^\star : S(\mathbf{1}) \to S(X)\otimes S(X^\vee)$.

Step 3. Consider now the subfunctor G_{id} of G that assigns to an algebra A assigns the set $\{(u,v) \in \mathrm{Hom}_A(A, A\otimes X) \times \mathrm{Hom}_A(A\otimes X, A) : v\circ u = \mathrm{id}_A\}$. We have that $c|_{G_{\mathrm{id}}}$ is a constant morphism. The functor $A \mapsto \{\mathrm{id}_A\}$ is representable by the algebra $\mathbf{1}$. So G_{id} is representable by the fibered product B defined by the following diagram

$$\begin{tikzcd} S(\mathbf{1}) \arrow[r, "\delta"] \arrow[d, "c^\star"'] & \mathbf{1} \arrow[d] \\ S(X)\otimes S(X^\vee) \arrow[r] & B \end{tikzcd}$$

Where $\delta : S(\mathbf{1}) \to \mathbf{1}$ is the map identifying each homogeneous component of $S(\mathbf{1})$ with $\mathbf{1}$. By definition, $\mathrm{Hom}_{\widetilde{\mathscr{C}}\text{-alg}^\beta}(B, B) = \{(u,v) \in \mathrm{Hom}_B(B, B\otimes X) \times \mathrm{Hom}_B(B\otimes X, B) : v\circ u = \mathrm{id}_B\}$. So $B\otimes X$ contains B as a direct summand. Let us say a bit more about the structure of the algebra B. First, by the construction of the fibered product, B is a quotient of $\mathbf{1}\otimes S(X)\otimes S(X^\vee) = S(X)\otimes S(X^\vee)$. We remark that the following diagram commutes:

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d, "\mathrm{coev}_X"'] & S(\mathbf{1}) \arrow[d, "c^\star"'] \\ X\otimes X^\vee \arrow[r] & S(X)\otimes S(X^\vee) \end{tikzcd}$$

where the top horizontal arrow is the map that identifies $\mathbf{1}$ with $S^1(\mathbf{1})$ and the bottom horizontal arrow identifies $X\otimes X^\vee$ with $S^1(X\otimes X^\vee)$ so that, by construction of the fibered product, B is the biggest quotient of $S(X)\otimes S(X^\vee) = S(X)\otimes S(X^\vee)$ that coequalizes coev_X and $c^\star|_{\mathbf{1}}$. Thus,

$$B \cong \bigoplus_{m\in\mathbb{Z}} \varinjlim_{n\geq 0} S^n(X)\otimes S^{n+m}(X^\vee)$$

where the transition map $S^n(X)\otimes S^{n+m}(X^\vee) \to S^{n+1}(X)\otimes S^{n+1+m}(X^\vee)$ is given by $\mathrm{coev}_X : \mathbf{1} \to X\otimes X^\vee$.

Step 4. We will show that B contains $\mathbf{1}$ as a direct summand, this will finish the proof. The pairing between $X^{\otimes n}$ and $(X^\vee)^{\otimes n}$ defines a pairing between $S^n(X)$ and $S^n(X^\vee)$, $\mathrm{ev}_n : S^n(X^\vee)\otimes S^n(X) \to \mathbf{1}$. Now set $\tau_0 = \mathrm{id}_{\mathbf{1}}, \tau_n = \mathrm{ev}_n / d_n$, where $d_n = \dim(S^n(X))$, here we are using that $\dim(X) \notin \mathbb{Z}_{<0}$ and so $d_n \neq 0$. It is easy to see that, these morphisms define a map

$$\varinjlim_{n\geq 0} S^n(X)\otimes S^n(X^\vee) \to \mathbf{1}$$

that splits the unit map. □

Remark 3.2. *We will use the following form of Lemma 3.1.* If $\dim(X) \in \mathbb{Z}_{>0} \subseteq \mathbb{K}$, then there exists a commutative $\widetilde{\mathscr{C}}$-algebra B such that the B-module $B\otimes X$ is isomorphic to $B^{\oplus \dim(X)} \oplus N$ as B-modules. Moreover, $B\otimes Y \neq 0$ for any object Y of $\widetilde{\mathscr{C}}$. *The proof goes exactly along the same lines as that of Lemma 3.1, only a small modification in Step 1 is required.*

Lemma 3.3. *Let $v : X \to Y$ be an epimorphism. Then, there exists an algebra $B \in \widetilde{\mathscr{C}}$-alg$^\beta$ (again, more precisely, in the ind-completion of $\widetilde{\mathscr{C}}$) such that the B-module morphism $\mathrm{id}_B \otimes v : B \otimes X \to B \otimes Y$ splits in the category of B-modules. Moreover, $B \otimes Z \neq 0$ for any object Z of $\mathscr{C}$.*

Proof. We proceed in two parts.

Step 1. $Y = \mathbf{1}$. Assume $Y = \mathbf{1}$, and let $v^\vee : \mathbf{1} \to X^\vee$ be the dual map. Let B be the fibered product defined by $\mathbf{1} \xleftarrow{\delta} S(\mathbf{1}) \xrightarrow{S(v^\vee)} S(X^\vee)$. By definition, the map $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^\vee$ defines a B-module map which is right inverse to $\mathrm{id}_B \otimes v$. Now we need to show that $B \otimes Z \neq 0$ for any object Z of $\mathscr{C}$. Consider a monomorphism $V \to W$. Since $\otimes$ is exact, we have that $V^{\otimes n} \to W^{\otimes n}$ is a monomorphism. So we can filter $S^n(W)$ with quotients being $S^i(V) \otimes S^{n-i}(W/V)$, $0 \leq i \leq n$. Applying this to the monomorphism $v^\vee : \mathbf{1} \to X^\vee$, we find that $S^n(M)$ is filtered with quotients being $S^i(X^\vee/\mathbf{1})$. This implies the statement about B.

Step 2. Y is arbitrary. Now take an arbitrary $v : X \to Y$. The functor $\underline{\mathrm{Hom}}(\bullet, Y) = \bullet^\vee \otimes Y$ is exact, so the map induced by v, $v' : \underline{\mathrm{Hom}}(X, Y) \to \underline{\mathrm{Hom}}(Y, Y)$ is injective. Now let X' be the fibered product of v' and the map $\mathrm{coev}_Y : \mathbf{1} \to \underline{\mathrm{Hom}}(Y, Y)$. Note that the projection $\pi : X' \to \mathbf{1}$ is an epimorphism. Now let B be the algebra constructed in Step 1. It is easy to see that, by construction, $\mathrm{id}_B \otimes v$ splits if and only if $\mathrm{id}_B \otimes \pi'$ splits. So we are done by Step 1. □

3.2. **Deligne's theorem.** We finish these notes with the following theorem, that was first proved by Deligne in [D]. We follow the proof of [R]. Recall the conventions on $\widetilde{\mathscr{C}}$ we have made at the beginning of this section.

Theorem 3.4. *The following are equivalent.*

(a) *$\widetilde{\mathscr{C}}$ is Tannakian, that is, there exist an algebra R and an exact, faithful, tensor functor $F : \widetilde{\mathscr{C}} \to R$-mod.*
(b) *For each $X \in \widetilde{\mathscr{C}}$, $\dim(X) \in \mathbb{Z}_{\geq 0}$.*
(c) *For each nonzero $X \in \widetilde{\mathscr{C}}$, $\dim(X) \in \mathbb{Z}_{>0}$.*
(d) *For each $X \in \widetilde{\mathscr{C}}$, there exists $n \in \mathbb{Z}_{>0}$ such that $\Lambda^n(X) = 0$.*

Proof. (a) ⇒ (b) is clear.

(c) ⇒ (b) is clear. For *(b)* ⇒ *(c)*, assume that there exists a nonzero object $X \in \widetilde{\mathscr{C}}$ with $\dim(X) = 0$. Since $X \neq 0$, the map $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^\vee$ is nonzero. If Y denotes the cokernel of this map, we get $\dim(Y) = -\dim \mathbf{1} = -1$, a contradiction. Hence, (b) and (c) are equivalent. It is clear that (c) and (d) are equivalent, too. So we have to show that (b), (c), (d) ⇒ (a). First, we remark that we can embed $\widetilde{\mathscr{C}}$ into its ind-completion. In particular, we will assume that $\widetilde{\mathscr{C}}$ has colimits. We remark that we can recover $\widetilde{\mathscr{C}}$ from its ind-completion $\widetilde{\mathscr{C}}^\wedge$: an object $X \in \widetilde{\mathscr{C}}^\wedge$ is in $\widetilde{\mathscr{C}}$ if and only if it is reflective.

Now let X be a nonzero object of $\widetilde{\mathscr{C}}$, let $d := \dim(X) \in \mathbb{Z}_{>0}$. By the proof of Lemma 3.1, there exists a commutative $\widetilde{\mathscr{C}}$-algebra R such that $R \otimes X \cong R^{\oplus d} \oplus N$. In particular, N is a direct summand in $\Lambda_R^{d+1}(R \otimes X) = R \otimes \Lambda^{d+1}(X) = 0$, so $N = 0$. Here Λ_R denotes the wedge product *in the category of R-modules*, which is a monoidal category by defining $X \otimes_R Y$ to be the coequalizer of the pair of maps:

$$\begin{tikzcd} X \otimes (R \otimes Y) \arrow[r, "\mathrm{id}_X \otimes m_Y"] \arrow[d, "\alpha_{X,R,Y}"'] & X \otimes Y \\ (X \otimes R) \otimes Y \arrow[r, "\beta_{X,R} \otimes \mathrm{id}_Y"] & (R \otimes X) \otimes Y \arrow[u, "m_X \otimes \mathrm{id}_Y"'] \end{tikzcd}$$

Taking direct limits over objects of $\widetilde{\mathscr{C}}$, there exists an algebra B_1 in $\widetilde{\mathscr{C}}$ such that $B_1 \otimes X$ contains $B^{\dim X}$ as a direct summand for any object $X \in \widetilde{\mathscr{C}}$. But now by Lemma 3.3, there exists an algebra $B \in \widetilde{\mathscr{C}}$ such that, for any exact sequence on B_1-mod, the sequence splits after tensoring with B. So the functor $X \mapsto \mathrm{Hom}_{\mathscr{C}}(\mathbf{1}, B \otimes X)$ is a fiber functor over the algebra $\mathrm{Hom}_{\widetilde{\mathscr{C}}}(\mathbf{1}, B)$.

□

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