RESEARCH STATEMENT

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My research focuses on representation theory, using techniques from algebraic geometry and noncommutative algebra. In the past, I have worked in general ring and module theory, around questions regarding relative injectivity and projectivity of modules over an associative ring, see [HLMS, LS]. My current research is more geometrically oriented. In a few words, I study the representation theory of algebras that quantize Poisson algebras. Mostly, I have studied rational Cherednik algebras, which are algebras associated to the action of a complex reflection group $W$ on a vector space $R$. In the next few pages, I intend to give an overview of my research and of the techniques I use to study the representation theory of these algebras.

1. Harish-Chandra bimodules for Rational Cherednik algebras

1.1. Set-up. Rational Cherednik algebras are a remarkable class of algebras defined by Etingof and Ginzburg in [EG]. They depend on the data of a complex reflection group $W$, its reflection representation $R$ and a conjugation invariant function $c : S \to \mathbb{C}$, where $S \subseteq W$ denotes the set of reflections. The rational Cherednik algebra $H_c := H_c(W, R)$ is a deformation of the smash-product algebra $\mathbb{C}[R \oplus R^*] \# W$. This means that it has a filtration with associated graded $\text{gr} \ H_c = \mathbb{C}[R \oplus R^*] \# W$. We remark that the algebra $H_c$ can be explicitly presented by generators and relations, cf. [EG (1.15)]. A consequence of this presentation that will be important for us is that $\mathbb{C}[R], \mathbb{C}[R^*]$ and $\mathbb{C} W$ sit inside $H_c$ as subalgebras. The representation theory of the rational Cherednik algebra - at $t = 1$, which we always assume, see e.g. [GGOR]- has many similarities to that of universal enveloping algebras. For example, one has a category $\mathcal{O}_c$ that was defined in [GGOR]. This is a highest-weight category whose set of standard modules is parametrized by irreducible representations of $W$. This category has been extensively studied in recent years, see e.g. [BE, GL, L5, L7, R, Sh, SV, We], and it is connected to the study of categorical Kac-Moody actions ([L7, Sh, SV, We]), to the geometry and combinatorics of Hilbert schemes ([GS, GS2]), and to categorical invariants of torus knots ([EGL, GORS]), to name just a few.

Of particular importance are functors between categories $\mathcal{O}$. One way to get such a functor is by tensoring with a special type of bimodules. More precisely, note that for two parameters $c, c' \in \mathbb{C}$ we have that the algebras $\mathbb{C}[R]^W, \mathbb{C}[R^*]^W$ are embedded in both $H_c$ and $H_{c'}$. The following definition appeared first in [BEG2].

Definition 1.1. A finitely generated $H_c$-$H_{c'}$-bimodule $B$ is said to be Harish-Chandra (shortly, HC) if, for every $a \in \mathbb{C}[R]^W \cup \mathbb{C}[R^*]^W$, the operator $\text{ad}(a) : b \mapsto ab - ba$ is locally nilpotent on $B$.

Since the algebras $H_c, H_{c'}$ are noetherian, the category $\text{HC}(c, c')$ of HC $H_c$-$H_{c'}$-bimodules forms an abelian subcategory of the category of all $H_c$-$H_{c'}$-bimodules. It is easy to show that $B \otimes_{H_c} N$ is in $\mathcal{O}_c$ for $B \in \text{HC}(c, c')$ and $N \in \mathcal{O}_{c'}$, and that $B_1 \otimes_{H_{c'}} B_2 \in \text{HC}(c, c'')$ for $B_1 \in \text{HC}(c, c')$, $B_2 \in \text{HC}(c', c'')$. In particular, $\text{HC}(c, c)$ is a monoidal category, with unit object the regular bimodule. It was shown by Ginzburg in [Gi, Corollary 6.3.3] that any HC bimodule has finite length, and results of Losev [L3, Theorem 3.4.6] imply that $\text{HC}(c, c')$ has finitely many simples and, moreover, that hom spaces are finite-dimensional. We have proved that the category $\text{HC}(c, c')$ has enough injectives, [Si, Remark 6.13] thus having.

Theorem 1.2. The category $\text{HC}(c, c')$ is equivalent to the category of representations of a finite-dimensional algebra.
The following questions naturally arise.

(Q1) What are the irreducible objects in \( \text{HC}(c, c') \)?

(Q2) Can we describe the monoidal structure on \( \text{HC}(c, c') \)?

(Q3) Can we describe the action of \( \text{HC} \)-bimodules on category \( \mathcal{O} \)?

My research has revolved around these questions. Next, I intend to give an overview of the main techniques I have used to approach these problems, and of the main results obtained in these directions.

1.2. Filtration by supports. A general approach that is used in the study of categories of \( \text{HC} \)-bimodules is to use a filtration of the category \( \text{HC}(c, c') \) by the (singular) support of a bimodule, see e.g. [G2, L, L3, O]. In order to do so, we need the following result due to Losev, [L3, Section 5.4]. Recall that the algebras \( H_c, H_{c'} \) are filtered and their associated graded is identified with the smash-product algebra \( \mathbb{C}[R \oplus R^*] W \), which is finite over its center, \( Z := \mathbb{C}[R \oplus R^*]^W \).

**Lemma 1.3.** A \( H_c - H_{c'} \)-bimodule \( B \) is \( \text{HC} \) if and only if it admits a bimodule filtration such that \( \text{gr} B \) is a finitely generated \( Z \)-module, that is, the left and right actions of \( Z \) on \( \text{gr} B \) coincide. We call a filtration satisfying this a good filtration on \( B \).

Thanks to this lemma, \( \text{HC} \) bimodules for rational Cherednik algebras may be viewed as a direct analogue of the corresponding notion for semisimple Lie algebras, see e.g. [BeGe]. Using Lemma 1.3 we may define the singular support of a \( \text{HC} \) bimodule \( B \), \( \text{SS}(B) \subseteq (R \oplus R^*) W \) to be the set-theoretic support of the \( Z \)-module \( \text{gr} B \) for a good filtration on \( B \). As usual, \( \text{gr} B \) depends on the choice of a good filtration but its support does not. It is immediate to see that \( \text{SS}(B) \) is a Poisson subvariety of the Poisson variety \( X := (R \oplus R^*) W \). In particular, it is a union of symplectic leaves. So for a symplectic leaf \( \mathcal{L} \subseteq X \), we may define the full subcategory \( \text{HC}_{\mathcal{L}}(c, c') \subseteq \text{HC}(c, c') \) consisting of those bimodules \( B \) with \( \text{SS}(B) \subseteq \overline{\mathcal{L}} \). Similarly, we may define \( \text{HC}_{\partial \mathcal{L}}(c, c') \), where \( \partial \mathcal{L} := \overline{\mathcal{L}} \setminus \mathcal{L} \). We remark that \( \text{HC}_{\partial \mathcal{L}}(c, c') \) is a Serre subcategory of \( \text{HC}_{\overline{\mathcal{L}}}(c, c') \). So we may define the quotient category \( \text{HC}_{\overline{\mathcal{L}}}(c, c') := \text{HC}_{\overline{\mathcal{L}}}(c, c') / \text{HC}_{\partial \mathcal{L}}(c, c') \) and

\[
\text{gr} \text{HC}(c, c') := \bigoplus_{\mathcal{L}} \text{HC}_{\mathcal{L}}(c, c')
\]

We remark that the variety \( X \) has finitely many symplectic leaves which are indexed by conjugacy classes of parabolic subgroups of \( W \), see [BrGa, Proposition 7.4], so the direct sum above is finite. We also remark that if \( c = c' \), the categories \( \text{HC}_{\mathcal{L}}(c, c) \) have a natural monoidal structure induced from that on \( \text{HC}(c, c) \). The filtration by supports is related to question (Q1)-(Q3) above as follows.

1. There is a bijection between the irreducibles in \( \text{HC}(c, c') \) and those in \( \text{gr} \text{HC}(c, c') \), see e.g. [G1, Section 5].

2. For irreducible bimodules \( B_1, B_2 \in \text{HC}(c, c') \) we have \( B_1 \otimes_{H_c} B_2 = 0 \) unless \( \text{SS}(B_1) = \text{SS}(B_2) \). This is in [Si, Corollary 2.6].

3. If \( B \in \text{HC}(c, c') \) and \( M \in \mathcal{O}_{c'} \) are irreducible, then \( B \otimes_{H_{c'}} M = 0 \) unless \( \pi(\text{SS}(B)) = \text{supp}(M) \).

Thus, the structure of the category \( \text{gr} \text{HC} \) gives us a good approximation to that of \( \text{HC} \).

1.3. Bimodules with full support. The variety \( X \) has a dense symplectic leaf, say \( \mathcal{L}_0 \). Let us denote \( \overline{\text{HC}}(c, c') := \text{HC}_{\mathcal{L}_0}(c, c') \). In other words, \( \overline{\text{HC}}(c, c') \) is the quotient of the category of all \( \text{HC} \)-\( H_{c', c'} \)-bimodules by the full subcategory of bimodules with proper support. Let us remark that, for \( c \) and \( c' \) outside of a countable collection of hyperplanes of the parameter space, we have that \( \overline{\text{HC}}(c, c') = \text{gr} \text{HC}(c, c') = \text{HC}(c, c') \).

In [BEG2, Theorem 8.16], it is shown that when \( W \) is a Coxeter group and \( c(s), c'(s) \in \mathbb{Z} \) for all \( s \in S \), then there is an equivalence \( \text{HC}(c, c') = \overline{\text{HC}}(c, c') \cong W\text{-rep} \), which is an equivalence of monoidal categories when \( c = c' \). This is done by looking at the action of the algebra \( H_c \) in the
space of differential operators on the space of c-quasi-invariants, see [BEG2] Section 7. Further work in [S2], after some small generalizations made in [S1], shows that when W is a Coxeter group then \( \text{HC}(c, c) \cong (W/W_c) \text{-rep} \), where \( W_c := \{ s \in S : c(s) \notin \mathbb{Z} \} \). The proof of this is based on the study of \( D \)-modules on \( R^e \mathfrak{g}/W \), which are related to modules over the Cherednik algebra via the Knizhnik-Zamolodchikov (KZ) functor, see [GGOR] Section 5.

For general complex reflection groups, we explicitly construct in [S1], Section 5] a normal subgroup \( W_c \subseteq W \) that coincides with the subgroup \( W_c \) described in the paragraph above when \( W \) is a Coxeter group. Let us denote by \( \mathfrak{p}_Z \) the space of parameters \( c \) for which \( W_c = \{ 1 \} \). This is a Z-lattice inside the space of all parameters, and it coincides with the lattice considered in [BC] L6. The subgroup \( W_c \) satisfies the following properties.

- \( W_c \) is a complex reflection group.
- If \( c \) is outside a countable collection of hypersurfaces, \( W_c = W \).
- \( W_c = W' \) provided there exists a character \( \varepsilon : W \to \mathbb{C}^\times \) with \( \varepsilon c - c' \in \mathfrak{p}_Z \).

We have then the following result, generalizing [BEG2], [Sp].

**Theorem 1.4 ([S1], Theorem 1.1).** The following is true.

1. The category \( \text{HC}(c, c') \) is nonzero if and only if there exists a character \( \varepsilon : W \to \mathbb{C}^\times \) with \( \varepsilon c - c' \in \mathfrak{p}_Z \).

2. Assume \( \text{HC}(c, c') \neq 0 \). Then, all categories \( \text{HC}(c, c'), \text{HC}(c', c), \text{HC}(c, c) \) and \( \text{HC}(c', c') \) are equivalent and they are equivalent to the category of representations of the group \( W/W_c \). For \( c' = c \), this is an equivalence of monoidal categories.

The proof of this theorem is based on translation functors introduced in [BC]. Theorem 1.4 gives an answer for (Q1)-(Q2) for the category \( \text{HC} \). An answer for (Q3) can be found by means of the KZ functor mentioned above, see [S1] Section 4).

1.4. **Type A.** Now we focus on type A, that is \( W = S_n \), the symmetric group on \( n \) elements. Here, there is a single conjugacy class of reflections, so we may think of the parameter \( c \) as a single complex number. We have recently proved the following result, [S2], which relates the category of HC bimodules with category \( \mathcal{O} \).

**Theorem 1.5.** Assume \( c \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0} \), and let \( H_c \) be the rational Cherednik algebra associated to the symmetric group \( S_n \) and parameter \( c \). Then, the functor \( \text{HC}(c, c) \to \mathcal{O}_c, B \mapsto B \otimes_{H_c} \Delta_c(\text{triv}) \) satisfies the following properties.

1. It is a fully faithful embedding.
2. Its image is closed under taking subquotients.
3. It is compatible with supports: for a parabolic subgroup \( W' \subseteq W \), the symplectic leaf \( \mathcal{L}_{W'} \) defined by \( W' \) is contained in \( \text{SS}(B) \) if and only if the set \( WR_{W'}/W \) is contained in \( \text{supp}(B \otimes_{H_c} \Delta_c(\text{triv})) \).

If \( c \in \mathbb{Q}_{< 0} \), a similar statement holds when substituting the trivial representation \( \text{triv} \) with the sign representation of \( S_n \).

We remark that the image of the functor in Theorem 1.5 is in general not closed under extensions. However, we can describe the simples in its image. We restrict to the case \( c = r/m \), where \( c > 0 \), \( 1 < m \leq n \) and \( \gcd(r; m) = 1 \), all other cases can be either reduced to this or to Theorem 1.4. Recall that the irreducibles in \( \mathcal{O}_c \) are parametrized by partitions \( \lambda \vdash n \). It is easy to see that for a partition \( \lambda \) there exists a unique decomposition \( \lambda = m\mu + \nu \) where \( \nu \) is \( m \)-regular, that is, \( \nu_i - \nu_{i-1} < m \) for every \( i \).

**Theorem 1.6.** Assume \( c = r/m > 0 \), \( \gcd(r; m) = 1 \) and \( 1 < m \leq n \). Let \( \lambda \vdash n \) be a partition, and decompose \( \lambda = m\mu + \nu \), where \( \nu \) is \( m \)-regular. Then, the irreducible \( L(\lambda) \in \mathcal{O}_c \) belongs to the image of the functor in Theorem 1.5 if and only if \( \nu \) is the image of the sign partition \( (1^{|\nu|}) \) of \( |\nu| \) under the Mullineux involution.
The proof of Theorem 1.6 is based on the theory of partial KZ functors, see e.g. [Wi]. We remark that the fact that $\nu$ is Mullineux-dual to $(1^\nu)$ is equivalent to requiring that the quotient of the Specht module $S(\nu)$ of the Iwahori-Hecke algebra $H_{\exp(2\pi\sqrt{c}T)}(\nu)$ by the radical of the symmetric form given by the cellular structure on $H_{\exp(2\pi\sqrt{c}T)}(\nu)$ is precisely the trivial representation of the Iwahori-Hecke algebra, i.e. the 1-dimensional representation where all the $T_i$’s act by 1.

The requirement that $c > 0$ is not essential: if $c < 0$, we have an isomorphism $H_c \to H_{-c}$ that identifies the categories of HC bimodules, the categories $O$, and intertwines the corresponding embeddings. We remark that a description of the categories $HC(c,c)$ follows easily from Theorem 1.6 and that we also have a description of the categories $HC_L(c,c')$ for different parameters $c,c'$. Since the statements are a bit technical, we refer to [Si, Theorem 1.2] for a precise formulation.

1.5. Future directions.

1.5.1. Other types. We remark that Theorem 1.5 does not generalize to arbitrary complex reflection groups and arbitrary parameters. The reason is that, in general, rational Cherednik algebras may have many finite-dimensional representations. For example, for $W = \mathbb{Z}/\ell\mathbb{Z}$, category $O_c$ always has $\ell$ irreducible objects, while $HC(c,c)$ may have as many as $(\ell - 1)^2 + 1$ irreducibles so, when this happens, there cannot be an embedding $HC(c,c) \to O_c$ with the properties of that in Theorem 1.5. Nevertheless if we restrict the space of parameters we do have a similar result. For example, if $O_c$ is semisimple, Theorem 1.3 implies that an analogue of Theorem 1.5 is valid.

However, the most interesting parameters are the ones where category $O_c$ is not semisimple. When $W$ is a real reflection group and the function $c$ is constant, $O_c$ fails to be semisimple precisely when $c$ is a rational number with denominator dividing one of the degrees of $W$, [DJO].

Conjecture 1.1. Assume $W$ is a Weyl group of type $D_n$. Then, an analogue of Theorem 1.5 holds, provided $c = r/d > 0$, with $d > n$ is an irreducible fraction. In particular, in this case we have a bijection between the set of two-sided ideals of $H_c$ and that of submodules of $\Delta(\text{triv})$.

The reason why Conjecture 1.1 should hold is that the Cherednik algebras with parameters as in the statement of the conjecture have few finite-dimensional representations. In particular, we conjecture that such an algebra does not have finite-dimensional representations unless $d = 2n - 1$ is the Coxeter number of $W$, in which case the only finite-dimensional representation is the irreducible quotient of $\Delta(\text{triv})$ (if $c > 0$) or of $\Delta(\text{sign})$ (if $c < 0$). When $d = 2$ this is no longer the case and an analogue of Theorem 1.5 is unlikely to hold.

1.5.2. Cyclotomic rational Cherednik algebras. One would like to have more information for bimodules with proper support for more general rational Cherednik algebras. A good starting point is the wreath-product group $G(\ell,1,n) = S_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})^n$ acting on $R = \mathbb{C}^n$. The category $O$ for this algebra has been extensively studied since it admits, via induction and restriction functors of [BE], a categorical action of an appropriate Kac-Moody algebra (that depends on the parameter $c$) see [GL, GM, Si, SV, L5, L7, RSVV, We]. Using these techniques and the theory of partial KZ functors, [GM], it is possible to describe the associated graded of the category $O$ under the support filtration, [SV, L7]. The corresponding subquotients of HC act on these subquotients. Similarly to the type A case, the study of these actions should give information about the category $HC_L(c,c)$, as follows. First of all, every irreducible HC bimodule is contained in a bimodule of the form $\text{Hom}_C(M,N)$, where $M, N \in O_c$ are irreducible modules with the same support. One problem here is that the bimodule $\text{Hom}_C(M,N)$ may not contain nonzero HC bimodules. This is where partial KZ functors come to play. To explain our idea, let us introduce some notation. The bimodule $\text{Hom}_C(M,N)$ contains a biggest HC subbimodule, which we denote by $F(M,N)$, see e.g. [L3, Proposition 5.7.1]. So we can study the modules $F(M,N) \otimes_{H_c} K \in O_c$ where $K$ is another irreducible module in category $O_c$. We remark that it suffices to study the case $\text{supp}(M) = \text{supp}(N) = \text{supp}(K)$, see [L3, Lemma 5.7.2] and [Si, Lemma 2.5]. We assume that this support has the form $WR' \setminus W'$ for some parabolic subgroup $W' \subseteq W$, and denote $KZ_{W'}$ for the corresponding partial KZ functor. We plan to study the image
of $\text{KZ}_W(f(M,N) \otimes_H K)$ in terms of $\text{KZ}_W(M), \text{KZ}_W(N)$ and $\text{KZ}_W(K)$. Of course, this technique works for any rational Cherednik algebra, but the case $W = G(1,1,n)$ is better understood. In particular, it is known that the image of the partial KZ functors is similar to the category of modules over a Hecke-type algebra for the group $N_W(W')/W'$ (which does not need to be a complex reflection group). Using explicit parameters for this Hecke-type algebra, we plan to obtain an upper bound on the number of irreducible HC bimodules with a given support, and use the techniques developed in [Si] to verify whether this bound is sharp.

1.5.3. Quantized Kleinian singularities. Harish-Chandra bimodules can be defined in greater generality. Namely, let $A$, $A'$ be filtered algebras quantizing the same graded Poisson algebra $A$. Then, a $A,A'$-bimodule $B$ is HC if it admits a bimodule filtration such that $\text{gr } B$ is a finitely generated $A$-module, i.e. the left and right actions of $A$ on $\text{gr } B$ coincide, cf. [BL, BPW]. This notion generalizes that of HC bimodules for (spherical) rational Cherednik algebras, for finite W-algebras, or for universal enveloping algebras.

A particularly important class of algebras are global sections of quantizations of conical symplectic resolutions, see e.g. [BPW, BLPW]. These generalize all three classes of algebras mentioned above, see [KR, L, BB]. Among these, perhaps the most basic ones are given by the minimal resolution of Kleinian singularities, i.e. quotients of $C^2$ by a finite subgroup of $\text{SL}_2(C)$. The corresponding algebra $A$ is a spherical symplectic reflection algebra, $\mathbb{E}_C$, and these are the spherical symplectic reflection algebras of smaller rank. We expect to, similarly to the RCA case, partially reduce the study of fully supported HC bimodules for general symplectic reflection algebras to these rank-2 algebras. In particular, we expect the following result to hold.

**Conjecture 1.2.** Let $G$ be a symplectic reflection group, with $S \subseteq G$ the set of symplectic reflections. Let $c : S \to C$ be a conjugation-invariant function, and consider the symplectic reflection algebra $H_c$. Then, there exists a normal subgroup $G_c \subseteq G$ satisfying the following properties.

1. $G_c$ is a symplectic reflection group.
2. $G_c = \{1\}$ if and only if $c$ is integral in the sense of [L6, Section 7]. For $c$ outside of a countable collection of hypersurfaces, $G_c = G$.
3. $G_c = G_{c'}$ provided $c - c'$ is integral.
4. There is an equivalence of monoidal categories, $\text{HC}(c, c) \cong (G/G_c)\text{-rep}$.

Moreover, $G_c = \langle G_{c'} : c' \subseteq G \text{ is a minimal nontrivial parabolic subgroup} \rangle$.

We remark that minimal parabolic subgroups as in the statement of Conjecture 1.2 are finite subgroups of $\text{SL}_2(C)$.

1.5.4. Quantized quiver varieties. Now let $X$ be a smooth Nakajima quiver variety, and $X_0$ its affinization, see e.g. [N]. This includes, in particular, the Kleinian singularities and the symmetric products $(C^2)^n/S_n$. We can consider quantizations of $X$, which are sheaves on the conical topology on $X$ and are parametrized by $\text{H}^2_{\text{DR}}(X, C)$, see [BK]. For a quantization with parameter $\lambda$, we let $A_{\lambda}$ be the algebra of global sections, this is a quantization of $C[X_0]$. Note that we have localization and global sections functors, see e.g. [BPW]. We have recently proved that, for special parameters of quantizations $\lambda \in \text{H}^2_{\text{DR}}(X, C)$ (those for which abelian localization holds) there is a duality functor $\text{HC}(\lambda, \lambda) \to \text{HC}(-\lambda + n\rho, -\lambda + n\rho)\text{opp}$. Here, $\rho$ is the Chern class of an ample line bundle on $X$ and $n$ is a sufficiently large integer. To produce this duality, we have to pass by categories of “twisted” Harish-Chandra bimodules. The twist here is by a graded, Poisson automorphism $f$ of $C[X_0]$, and twisted just means that $B$ admits a filtration such that $\text{gr } B$ is finitely generated and the right action of $C[X_0]$ on $\text{gr } B$ coincides with the twist of the left action by the automorphism $f$. In particular, when $f$ is the identity we recover the usual category of HC bimodules.

**Conjecture 1.3.** The duality functor commutes with the restriction functor introduced in [BL].

To prove Conjecture 1.3 we plan produce restriction functors for the categories of twisted HC bimodules. This will require studying the automorphisms $f$ locally and their behavior on slices.
1.5.5. Quantum Hamiltonian reductions. Among conical symplectic resolutions, we can take those that can be obtained by Hamiltonian reduction. These are spaces $X$ associated to a vector space $E$ and a reductive group acting linearly on it, and so acting in a Hamiltonian way on $T^*E$, see \[BPW\] \[L2\] for details. In this case, we can get quantizations of $\mathcal{D}_\lambda$ of $X$ from the sheaf $\mathcal{D}_E$ of microlocal differential operators on $E$, see \[BPW\] \[BL\] \[L2\]. In this setting, we get a Hamiltonian reduction functor from $\mathcal{D}_E$-modules to $\mathcal{D}_\lambda$-modules, see \[BPW\] \[KR\]. Doubling the constructions, we get a functor from $\mathcal{D}_{E \times E}$-modules to $\mathcal{D}_\lambda \mathcal{D}_\nu$-bimodules, where $\lambda, \nu \in (\mathfrak{g}^*)^G$. A natural question, then, is to find a suitable category of $\mathcal{D}_{E \times E}$-modules whose image under the Hamiltonian reduction functor is HC. We have found that this is the category of $(\mathfrak{g} \times \mathfrak{g})$-equivariant (in a suitable sense) $\mathcal{D}_{E \times E}$-modules admitting a filtration whose associated graded is killed by the ideal of the variety $\mu^{-1}(0) \times_{X_0} \mu^{-1}(0)$. We remark that such a $\mathcal{D}_{E \times E}$-module has to be holonomic. This implies, for example, that any HC bimodule has finite length. Note that there is also a Hamiltonian reduction functor at the algebraic level. This takes $\mathcal{D}_E$-modules to $\mathcal{A}_\lambda$-modules, where $\mathcal{A}_\lambda := [\mathcal{D}_E / \mathcal{D}_E (\Phi(\xi) - \lambda(\xi) : \xi \in \mathfrak{g})]^G$. Here, $\mathcal{D}_E$ is the algebra of global differential operators on $E$. We remark that several interesting algebras are of this form. For example, quantizations of quiver varieties arise as quantum Hamiltonian reductions, and special cases of these are the quantizations of Kleinian singularities discussed above.

We plan to study the variety $\mu^{-1}(0) \times_{X_0} \mu^{-1}(0)$, which should give information about Harish-Chandra bimodules. For example, knowing the components of $\mu^{-1}(0) \times_{X_0} \mu^{-1}(0)$ it should be possible to classify, similarly to \[BeGi\, Theorem 1.5.2\], the possible supports of HC bimodules, at least for special parameters $\lambda, \mu$. We also plan to study, via the Riemann-Hilbert correspondence, regular holonomic $\mathcal{D}_{E \times E}$-modules supported on this Steinberg-type variety. By inspecting conditions on which such a module admits a $(\mathfrak{g} \times \mathfrak{g})$-equivariant structure, this will produce HC $\mathcal{D}_\lambda \mathcal{D}_\mu$-bimodules.

2. Relative injectivity and projectivity from a lattice-theoretic point of view.

2.1. Relative injectivity and projectivity. My previous research lies in the theory of general rings and modules. So let $A$ be an associative ring with unity. From now on, the word module will always mean left module, unless explicitly stated otherwise. Let $M$ and $N$ be $A$-modules. The module $M$ is said to be $N$-injective if the functor $\text{Hom}_A(\bullet, M)$ is exact at all short exact sequences having $N$ in the middle. This notion of relative injectivity was introduced to study questions regarding cancellations and exchange properties of a module, see e.g. \[MM\]. Given a module $M$, we can define its injectivity domain, $\text{I}_n(M) := \{N \in A\text{-Mod} : M \text{ is } N\text{-injective}\}$. Of course, it follows from the definition that a module $M$ is injective if and only if $\text{I}_n(M) = A\text{-Mod}$. This leads to the question of which collections of modules can appear as the injectivity domain of a module. In \[LS\], we solve this problem as follows.

**Theorem 2.1 (\[LS\, Theorem 2.9\).** Let $C \subseteq A\text{-Mod}$. The following are equivalent.

1. $C$ contains all semisimple $A$-modules and it is closed under subquotients and arbitrary direct sums.
2. There exists an $A$-module $M$ with $C = \text{I}_n(M)$.

Now let us define $\text{I}(A)$ to be the collection of all classes of modules which appear as the injectivity domain of a module. In particular, $\text{I}(A)$ is a complete lattice. We remark that Theorem 2.1 implies that the lattice $\text{I}(A)$ is modular and equivalent to an interval in several lattices of importance in torsion theory, see e.g. \[BKN\] \[Gol\] \[Gol2\] \[St\]. Of course, one can consider analogous definitions for projectivity. A module $M$ is $N$-projective if $\text{Hom}_A(M, \bullet)$ is exact at every short exact sequence having $N$ in the middle. As before, we can define $\text{P}_n^{-1}(M) := \{N \in A\text{-Mod} : M \text{ is } N\text{-projective}\}$, and a class $\text{P}(A) := \{\text{P}_n^{-1}(M) : M \in A\text{-Mod}\}$ which, modulo some obvious set theory, is a semilattice. The study of $\text{P}(A)$ is, however, considerably harder than that of $\text{I}(A)$. For example, even the question of whether $\text{P}(A)$ is cardinalite leads to foundational problems and there is strong evidence that it may be undecidable in ZFC, \[LS\, Theorem 4.2\]. However, $\text{P}(A)$ is manageable for those rings $A$ for which projectivity is well-behaved. Namely, recall that a ring $A$ is called left perfect if every left $A$-module has a projective cover. For example, a (left or right) artinian ring is (left and right) perfect.
Theorem 2.2 ([LS], Proposition 4.7 and Corollary 4.10). Let $A$ be a left perfect ring. Then, $\mathfrak{p}_D(A)$ is a lattice and it is anti-isomorphic to the lattice of two-sided ideals that are contained in the Jacobson radical $J(A)$. If, moreover, $A$ is left artinian, then $\mathfrak{p}_D(A) = \mathfrak{i}_D(A)$.

2.2. A special class of QF rings. From Theorem 2.2 we see that, if $A$ is a left artinian ring, then for every $A$-module $M$ we can find an $A$-module $N$ with $\mathfrak{N}^{-1}(M) = \mathfrak{J}^{-1}(N)$. Obviously, in general we cannot take $M = N$, but in some cases we can indeed do this. Take, for example, $A = k[x]/(x^n)$ for any field $k$. It turns out that such rings are the prototype for this phenomena. To state our next theorem, we need to recall a few concepts. First, a ring $A$ for any field $K$ we cannot take $A = \mathfrak{M}_A$ for every $A$.

A special class of QF rings.

2.2. (QF) if every projective $A$-module $P$ is isomorphic with a power of the Jacobson radical. Now recall that a ring $A$ is said to be left uniserial if its set of left ideals is linearly ordered. If $A$ is, moreover, left artinian, then any (left, right, or two-sided) ideal of $A$ coincides with a power of the Jacobson radical. Now recall that a ring $A$ is said to be quasi-Frobenius (QF) if every projective $A$-module is injective. For example, a Frobenius algebra is QF.

Theorem 2.3 ([LS], Theorem 5.6). For any ring $A$, the following are equivalent.

1. For any left $A$-module $M$, we have $\mathfrak{N}^{-1}(M) = \mathfrak{P}^{-1}(M)$.
2. For any right $A$-module $M$, we have $\mathfrak{N}^{-1}(M) = \mathfrak{P}^{-1}(M)$.
3. For any two-sided ideal $I$ of $A$, the factor ring $A/I$ is QF.
4. $A \cong \prod_{i=1}^n \text{Mat}_{m_i}(D_i)$, where $m_i \in \mathbb{Z}_{>0}$ and $D_1, \ldots, D_n$ are uniserial artinian rings.

We remark that left-right symmetric statements of the form (1) $\iff$ (2) in the theorem above are important in ring theory, see e.g. [II, III]. The following corollary illustrates the usefulness of Theorem 2.3.

Corollary 2.4 ([LS], Proposition 5.7). Let $A$ be a QF ring. Then, the following are equivalent.

1. $A/\text{Soc}(A)$ is a simple ring.
2. $J(A)$ does not properly contain nonzero two-sided ideals.
3. $J(A)$ is semisimple as a left (right) $A$-module, and any two simple subquotients of $J(A)$ are isomorphic.
4. $A \cong S \times T$, where $S \cong \text{Mat}_n(D)$ for an uniserial artinian ring of length 2 $D$, and $T$ is semisimple artinian.

2.3. Future directions. While Theorem 2.1 is theoretically satisfactory, its proof does not give a way to actually construct a module $M$ with $\mathfrak{N}^{-1}(M) = \mathfrak{C}$. Of course, a method for doing this in complete generality is out of reach. But for some classes of rings methods could be available. For example, if $A$ is left artinian then, if we denote by $E$ the direct sum of the injective hulls of simple modules, it is not hard to see that $\mathfrak{C} = \mathfrak{N}^{-1}(M)$, where $M$ is the trace of $\mathfrak{C}$ on $E$. Matlis theory on injective modules could lead us to believe that this can be generalized to the case where $R$ is left noetherian. This is, however, not the case: the class of torsion abelian groups is not the injectivity domain of any torsion abelian group, so the method of taking traces does not work. Instead, we conjecture that a correct generalization of this statement is to take $A$ to be a left semiartinian ring, that is, the socle of a nonzero module is nonzero.

Conjecture 2.1. Let $A$ be a left semiartinian ring, and let $E$ be a main injective module in the sense of [RRRP]. Let $\mathfrak{C} \in \mathfrak{i}_D(A)$, and let $M$ be the trace of $\mathfrak{C}$ on $E$. Then $\mathfrak{C} = \mathfrak{N}^{-1}(M)$.

We remark that, for projectivity domains (assuming the ring is left perfect) an analogous question is very simple, see [LS, Lemma 4.6].

References


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