1 (Twisted) Differential operators and $\mathcal{D}$-modules.

1.1 Differential Operators.

Let $X$ be an algebraic variety. In this text, we assume all algebraic varieties are over $\mathbb{C}$. Let $\mathcal{R}$ be a sheaf of rings on $X$. For $\mathcal{R}$-modules $\mathcal{F}$ and $\mathcal{G}$, define the sheaf $\text{Hom}_\mathcal{R}(\mathcal{F}, \mathcal{G})$ as follows. For any open set $U \subset X$, the sections of $\text{Hom}_\mathcal{R}(\mathcal{F}, \mathcal{G})$ on $U$ are $\text{Hom}_{\mathcal{R}|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$. For any algebraic variety $X$, let $\mathbb{C}_X$ be the constant sheaf of complex numbers $\mathbb{C}$.

Now assume that $X$ is a smooth complex algebraic variety. Let $\mathcal{O}_X$ be its sheaf of regular functions and $\mathcal{V}_X$ be its sheaf of vector fields, that is, $\mathcal{V}_X := \text{Der}_\mathbb{C}(\mathcal{O}_X) := \{ \theta \in \mathcal{E}_{\mathcal{O}_X}(\mathcal{O}_X) : \theta(\phi f) = f \theta(\phi) + \theta(f) \phi, f, \phi \in \mathcal{O}_X \}$. The sheaf $\mathcal{D}_X$ of differential operators on $X$, is, by definition, the subsheaf (of associative rings) of $\mathcal{E}_{\mathcal{O}_X}(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ and $\mathcal{V}_X$.

**Theorem 1.1** In the setup of the previous paragraph, if $\dim X = n$, then for every point $p \in X$ there exists an affine open neighborhood $U$ of $p$, $x_1, \ldots, x_n \in \mathcal{O}_X(U)$ and $\partial_1, \ldots, \partial_n \in \mathcal{V}_X(U)$ such that $[\partial_i, \partial_j] = 0$, $\partial_i(x_j) = \delta_{ij}$ and $\mathcal{V}_X(U)$ is free over $\mathcal{O}_X(U)$ with basis $\partial_1, \ldots, \partial_n$.

**Proof.** Let $m_p$ be the maximal ideal of $\mathcal{O}_{X,p}$. Since $X$ is smooth of dimension $n$, there exist functions $x_1, \ldots, x_n$ generating $m_p$. Then, $dx_1, \ldots, dx_n$ is a basis of $\Omega_{X,p}$, where $\Omega_X$ is the cotangent sheaf of $X$. Take $\partial_1, \ldots, \partial_n \in \mathcal{V}_X$ to be the dual basis of $dx_1, \ldots, dx_n$. \qed

**Definition 1.2** If $x_1, \ldots, x_n$, $\partial_1, \ldots, \partial_n$ are as in Theorem 1.1, then we say that $\{x_i\}$ is a local coordinate system. Note that, since $dx_i$ are linearly independent, a local coordinate system determines a basis $\partial_1, \ldots, \partial_n$ of $\mathcal{V}_X(U)$. Thus, sometimes we also refer to $\{x_i, \partial_i\}$ as a local coordinate system.

**Remark 1.3** A local coordinate system does not give an isomorphism from $U$ to an affine subvariety of $\mathbb{C}^n$. We only have an étale morphism $U \to \mathbb{C}^n$.

**Theorem 1.4** Let $U$ be an affine open subset of $X$ and $\{x_i, \partial_i\}$ a local coordinate system on $U$. Then, any differential operator of order $\leq k$ on $U$ can be uniquely written in the form

$$\sum_{k_1 + \cdots + k_n \leq k} f_{k_1, \ldots, k_n} \partial_1^{k_1} \cdots \partial_n^{k_n},$$

where $f_{k_1, \ldots, k_n} \in \mathcal{O}_X(U)$.

Note that $\mathcal{D}_X$ can be described by generators and relations as follows: it is generated by $\mathcal{O}_X$ and $\mathcal{V}_X$ with relations:

- $f_1 \cdot f_2 = f_1 f_2$,
- $f \cdot \xi = f \xi$,
- $\xi \cdot f = \xi(f) + f \xi$,
- $\xi_1 \cdot \xi_2 - \xi_2 \cdot \xi_1 = [\xi_1, \xi_2]$,

where $f, f_1, f_2 \in \mathcal{O}_X$ and $\xi, \xi_1, \xi_2 \in \mathcal{V}_X$. 

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1.2 A filtration on $\mathcal{D}_X$.

Let $A$ be a commutative ring and let $M, N$ be $A$-modules. Define $A$-modules $\text{Diff}^\leq_{\leq 0}(M, N) := \text{Hom}_A(M, N)$ inductively by:

1. $\text{Diff}^\leq_{\leq 0}(M, N) := \text{Hom}_A(M, N)$.
2. $\text{Diff}^\leq_{\leq n+1}(M, N) := \{ \theta \in \text{Hom}_M(M, N) : [f, \theta] \in \text{Diff}^\leq_{\leq n}(M, N) \text{ for every } f \in A \}$.

Set $\text{Diff}(M, N) = \bigcup_n \text{Diff}^\leq_{\leq n}(M, N)$.

Remark 1.5 If, moreover, $A$ is a $k$-algebra, then we add the extra condition that every map inside $\text{Diff}^\leq_{\leq n}(M, N)$ is $k$-linear.

Definition 1.6 Let $M, N$ be $\mathcal{O}_X$ modules. Define $\mathcal{D}_X(\mathcal{M}, \mathcal{N})$ by gluing $\text{Diff}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ on affine open subsets, that is, for any affine open subset $U \subseteq X$,

$$\Gamma(U, \mathcal{D}_X(\mathcal{M}, \mathcal{N})) := \text{Diff}_{\mathcal{O}_U}(\mathcal{M}(U), \mathcal{N}(U)).$$

Note that $\mathcal{D}_X = \mathcal{D}_X(\mathcal{O}_X, \mathcal{O}_X)$. It follows that $\mathcal{D}_X$ is a filtered sheaf. In local coordinates $(U, \{x_i, \partial_i\})$, $F_j \mathcal{D}_X(U) = \bigoplus \mathcal{O}_U \partial_{x_i}^{\alpha_i} \partial_{x_j}^{\alpha_j} \ldots \partial_{x_k}^{\alpha_k} \delta_{\alpha_i, \alpha_j, \ldots, \alpha_k}$, that is, $F_j \mathcal{D}_X(U)$ is given by differential operators of order $\leq j$. Note that if $P \in F_j \mathcal{D}_X$, $Q \in F_m \mathcal{D}_X$, then $[P, Q] \in F_{j+m-1} \mathcal{D}_X$. Then, $\text{gr} \mathcal{D}_X$ is a sheaf of commutative algebras. Let us look closer at $\text{gr} \mathcal{D}_X$. Take a local coordinate system $(U, \{x_i, \partial_i\})$. Set $\xi_i = \partial_i \in F_1 \mathcal{D}_X(U)/F_0 \mathcal{D}_X(U)$. Then, $\text{gr} \mathcal{D}_X(U) = \mathcal{O}_U[\xi_1, \xi_2, \ldots, \xi_n]$.

Let $\pi : T^*X \to X$ be the projection. Regard $\xi_1, \ldots, \xi_n$ as the local coordinate system on the cotangent space $T^*_x X$. Then, $\mathcal{O}_U[\xi_1, \ldots, \xi_n]$ is identified with $\pi_* \mathcal{O}_{T^*X}(U)$. These identifications are natural and they can be glued together, so that we have an isomorphism,

$$\text{gr} \mathcal{D}_X \cong \pi_* \mathcal{O}_{T^*X}.$$

A natural isomorphism can be constructed as follows. First of all, note that $F_0 \mathcal{D}_X = \mathcal{O}_X$. This follows easily by the description of $\mathcal{D}_X$ in local coordinates. Alternatively, we have a monomorphism $\mathcal{O}_X \hookrightarrow F_0 \mathcal{D}_X$. An inverse map is given by $F_0 \mathcal{D}_X \ni P \mapsto P(1)$. Now we see that $F_1 \mathcal{D}_X = \mathcal{O}_X \oplus \mathcal{V}_X$. Note that, if $P \in F_1 \mathcal{D}_X$, then $f \mapsto [P, f]$ is a derivation on $\mathcal{O}_X$. Then, we get a morphism $F_1 \mathcal{D}_X \to \mathcal{O}_X \oplus \mathcal{V}_X$, $P \mapsto (P(1), [P, \bullet])$. This is an isomorphism.

So we have isomorphisms $\mathcal{O}_X = \text{gr}_0 \mathcal{D}_X$, $\mathcal{V}_X = \text{gr}_1 \mathcal{D}_X$. These give rise to a map $\pi_* \mathcal{O}_{T^*X} \to \text{gr} \mathcal{D}_X$. By the above considerations in local coordinates, this is an isomorphism.

1.3 Twisted differential operators.

Definition 1.7 Let $\mathcal{D}$ be a sheaf of rings on $X$ that admits an inclusion $\iota : \mathcal{O}_X \hookrightarrow \mathcal{D}$. We say that $\mathcal{D}$ is a sheaf of twisted differential operators (TDO) if the embedding $\mathcal{O}_X \hookrightarrow \mathcal{D}$ is locally isomorphic to the standard embedding $\mathcal{O}_X \hookrightarrow \mathcal{D}_X := \mathcal{D}_X(\mathcal{O}_X, \mathcal{O}_X)$.

For any line bundle $\mathcal{L}$ on $X$, $\mathcal{D}_X^\mathcal{L} = \mathcal{D}_X(\mathcal{L}, \mathcal{L})$ is a TDO, as $\mathcal{L}$ is locally isomorphic to $\mathcal{O}_X$.

One can also get a sheaf of TDO from a closed 1-cocycle $\alpha \in \Omega^1_{cl}$ as follows. Consider an open cover $X = \bigcup U_i$ and a 1-cocycle $\alpha = (\alpha_{ij}) \in \Omega^1_{cl}$. Then, $\mathcal{D}(U_i) := \mathcal{D}_X(U_i)$, and the transition function from $U_j$ to $U_i$ maps a vector field $\xi$ to $\xi + \langle \xi, \alpha_{ij} \rangle$. In fact, by this procedure we can get all TDO.

Proposition 1.8 Let $\varphi : \mathcal{D}_X \to \mathcal{D}_X$ be an endomorphism such that $\varphi|_{\mathcal{O}_X} = \text{id}$. Then, there exists $\omega \in \Omega^1_{cl}$ such that $\varphi(\theta) = \theta - \omega(\theta)$ for any vector field $\theta \in \mathcal{V}_X$. Moreover, $\varphi$ is completely determined by $\omega$ and it is an automorphism of $\mathcal{D}_X$.

Proof. Let $f \in \mathcal{O}_X$, $\theta \in \mathcal{V}_X$. Then, $[\varphi(\theta), f] = \varphi(\theta, f) = \varphi(\theta(f)) = \theta(f)$. Then $[\varphi(\theta), f](1) = \theta(f)(1)$, so $\varphi(\theta)(f) = \theta(f) + f \varphi(\theta)(1)$. Set $\omega(\theta) = -\varphi(\theta)(1)$. This is a 1-form. Note that $\omega(\theta, \eta) = -\varphi(\theta, \eta)(1) = -[\varphi(\theta), \varphi(\eta)](1) = \varphi(\theta)(\omega(\eta)) - \varphi(\eta)(\omega(\theta)) = \theta(\omega(\eta)) - \eta(\omega(\theta))$. It follows that $\omega$ is closed. The last statement of the Proposition is clear. $\square$

The following is then an exercise in Čech cohomology.
Proposition 1.9 Twisted differential operators on a smooth variety $X$ are classified by the first cohomology $H^1_{2\text{ar}}(X, \Omega^1_X)$.

To prove Proposition 1.9, one uses a covering $U_i$ of $X$ by open affine subsets such that the sheaf $\mathcal{D}$ is locally isomorphic to $\mathcal{D}_X$ in each $U_i$, and the transition morphisms in each intersection $U_i \cap U_j$.

Remark 1.10 If $\mathcal{L}$ is a line bundle, then we know that $\mathcal{D}_X^0$ is a TDO. The Picard group is naturally isomorphic to $H^1(X, O_X^*)$, where $O_X^*$ is the sheaf of invertible elements in $O_X$. There exists a homomorphism $O_X \to O_X^*$ given by taking the logarithmic derivative, $f \mapsto \text{dlog}(f) = f^{-1} df$, which induces morphisms $H^p(\text{dlog}) : H^p(X, O_X^*) \to H^p(X, O_X^*)$. It is an exercise to show that the 1-cocycle corresponding to $\mathcal{D}_X^0$ is $H^1(\text{dlog})(\mathcal{L})$.

Note that it follows that any TDO $\mathcal{D}$ is filtered, and moreover, $\text{gr} \mathcal{D} = \pi_* O_{T^*X}$.

1.4 Homogeneous TDO.

In this subsection, we assume that $X$ is a homogeneous $G$-variety, where $G$ is a semisimple algebraic group. Later, we will specialize the results of this subsection to the case $X = G/B$, where $B$ is a Borel subgroup of $G$. Let $\mathfrak{g} := \text{Lie}(G)$.

Recall that the universal enveloping algebra of $\mathfrak{g}$ is $\mathcal{U}(\mathfrak{g}) = T\mathfrak{g} / (u \otimes v - v \otimes u - [u, v], u, v \in \mathfrak{g})$.

Differentiating the action of $G$ on $O_X$, we get a $G$-equivariant Lie algebra homomorphism $\gamma : \mathfrak{g} \to \Gamma(X, V_X)$, that can be extented to a $G$-equivariant algebra homomorphism $\gamma : \mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X)$. More generally, we make the following definition.

Definition 1.11 let $\mathcal{D}$ be a TDO on $X$ with an algebraic action $\gamma$ of $G$ on $\mathcal{D}$ and a $G$-equivariant algebra homomorphism $\alpha : \mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathcal{D})$ satisfying the following conditions:

1. The multiplication in $\mathcal{D}$ is $G$-equivariant.

2. For $\xi \in \mathfrak{g}$, we have an equality $\xi_X = [\alpha(\xi), \bullet]$, where $\xi_X$ is the the derivation induced by differentiating the $G$-action on $\mathcal{D}$.

In this case, we say that $\alpha$ is a quantum momentum map and we call $(\mathcal{D}, \gamma, \alpha)$ a homogeneous sheaf of twisted differential operators (HTDO).

For example, $(\mathcal{D}_X, \gamma_X, \tau)$ is an HTDO, where $\gamma_X$ is the natural action of $G$ on $\mathcal{D}_X$. Our next goal is to classify HTDO. To do this, we study $(\mathcal{D}_X, \gamma_X, \tau)$ more closely.

Consider the trivial bundle $g \times X \to X$, and its sheaf of sections $g^0 := O_X \otimes_{\mathbb{C}} g$. We can define a bracket on $g^0$ by

$$[f \otimes \xi, g \otimes \eta] := f \tau(\xi) g \otimes \eta - g \tau(\eta) f \otimes \xi + f \otimes [\xi, \eta].$$

This makes $g^0$ a sheaf of Lie algebras. It is also an $O_X$-bimodule. Note, however, that the Lie bracket is not $O_X$-bilinear. We can extend $\tau : g \to \Gamma(X, V_X)$ to $\tau^0 : g^0 \to V_X$ by $\tau^0(f \otimes \xi) = f \tau(\xi)$. Note that this satisfies the following identity:

$$[f \otimes \xi, h(g \otimes \eta)] = \tau^0(f \otimes \xi)(h)(g \otimes \eta) + h[f \otimes \xi, g \otimes \eta],$$

where $f, g, h \in O_X$, $\xi, \eta \in \mathfrak{g}$.

Let $U^0$ be the sheaf of algebras generated by $g^0$ and $O_X$ subject to the following relations:

(a) $\iota(fg) = \iota(f) \cdot \iota(g)$.

(b) $\iota([a, b]) = \iota(a) \cdot \iota(b) - \iota(b) \cdot \iota(a)$.

(c) $\iota(fa) = \iota(f) \cdot \iota(a)$.

(d) $\iota(a) \cdot \iota(f) - \iota(f) \cdot \iota(a) = \iota(\tau^0(a)(f))$.

Where $f, g \in O_X$, $a, b \in g^0$.

Remark 1.12 In the formalism of Lie algebroids, $g^0$ is a Lie algebroid with $\tau^0 : g^0 \to V_X$ the anchor map (this follows from Equation 7), and $U^0$ is the universal enveloping algebra of $g^0$. 


Note that $\tau^\circ$ induces a map $\tau^\circ : U^\circ := U(\mathfrak{g}^\circ) \to D_X$. This map is an epimorphism. This follows from the fact that $D_X$ is generated by $V_X$ and $O_X$, and the following Proposition.

**Proposition 1.13** The morphism $\tau^\circ : \mathfrak{g}^\circ \to V_X$ is an epimorphism.

*Proof.* Since $\mathfrak{g}^\circ$ and $V_X$ are locally free, it suffices to show that the induced map on the geometric fibers of $\mathfrak{g}^\circ$ and $V_X$ is surjective. But this is clear. □

So $D_X = U^\circ / \text{Ker}(\tau^\circ)$. We now give a description of $\text{Ker}(\tau^\circ)$. Of course, first we need an explicit characterization of $U^\circ$.

**Proposition 1.14** As a sheaf, $U^\circ = O_X \otimes C U(\mathfrak{g})$. The product is given by

$$(f \otimes \xi)(g \otimes \eta) = f \tau(\xi)g \otimes \eta + fg \otimes \xi \eta.$$ 

Where $f, g \in O_X$, $\xi \in \mathfrak{g}$, $\eta \in U(\mathfrak{g})$.

*Proof.* Follows from the relations defining $U^\circ$. □

Note that it also follows that $U^\circ$ is a filtered algebra, by setting $F_p U^\circ := O_X \otimes C F_p U(\mathfrak{g})$, where $F_p U(\mathfrak{g})$ is the standard filtration on $U(\mathfrak{g})$. Note that $\tau^\circ$ is a filtered morphism, that $F_0 U^\circ = O_X$, $F_1 U^\circ = O_X \otimes \mathfrak{g}^\circ$ and that $U^\circ$ is generated by $F_1 U^\circ$ as a sheaf of algebras.

Let $b^\circ = \text{Ker}(\tau^\circ : \mathfrak{g}^\circ \to V_X)$, so that $b^\circ$ consists of those sections $\sum f_i \otimes \xi_i$ such that $\sum f_i \tau(\xi_i) = 0 \in V_X$. Note that $J_0 := b^\circ U^\circ$ is a sheaf of two-sided ideals in $U^\circ$: it is easy to see that, if $\sum f_i \otimes \xi_i \in b^\circ$ and $\eta \in \mathfrak{g}$, then $[1 \otimes \eta, \sum f_i \otimes \xi_i] \in b^\circ$. Similarly, if $g \in O_X$, then

$$\left[ \sum f_i \otimes \xi_i, g \otimes 1 \right] = \sum f_i \tau(\xi_i)g \otimes 1 = 0.$$

Alternatively, one can see that $J_0$ is a sheaf of two-sided ideals from the following Proposition.

**Proposition 1.15** $\text{Ker}(\tau^\circ : U^\circ \to D_X) = J_0$.

*Proof.* It is clear that $J_0$ is contained in $\text{Ker}(\tau)$. Moreover, for any $x \in X$, the geometric fiber $T_x(J_0) = b^\circ_0 U(\mathfrak{g})$ is the kernel of the induced map from the geometric fiber $T_x(U^\circ) = U(\mathfrak{g})$ to the geometric fiber of $D_X$ at $x$. The result follows. □

Then, we have that $D_X = U^\circ / J_0$.

Now we show that any HTDO admits a similar description. Let $(D, \gamma, \alpha)$ be an HTDO. Then, for every $\xi \in \mathfrak{g}$ and $f \in O_X$, $[\alpha(\xi), f] = \tau(\xi)f$. It follows that $\alpha : \mathfrak{g} \to F_1 D$. Note that we can also extend $\alpha$ to an algebra homomorphism $\alpha^\circ : U^\circ \to D$, $f \otimes \xi \mapsto f \alpha(\xi)$. Moreover, $\alpha^\circ$ is filtered and $\text{gr} \alpha^\circ = \text{gr} \tau^\circ$, so $\alpha^\circ(b^\circ) \subseteq F_0 D = O_X$. Then, an HTDO determines a $G$-equivariant morphism from the $G$-homogeneous $O_X$-module $b^\circ$ to $O_X$.

Fix a point $x_0 \in X$. Let $B_0 = \text{Stab}_X(x_0)$, and let $b_0$ be the fiber of $b^\circ$ at $x_0$. Then, $B_0$ acts on $b^\circ_0$. Let $I(b_0^\circ)$ be the space of $B_0$-invariants. There is a natural linear isomorphism between $I(b^\circ_0)$ and the space of $G$-equivariant morphisms $\sigma$ of $b^\circ$ to $O_X$. This follows from the correspondence between $G$-equivariant vector bundles on $X$ and representations of $B_0$, see e.g. [ Section 9.11]. Then, for each $\lambda \in I(b^\circ_0)$, let $\sigma_\lambda$ be the associated $G$-equivariant morphism $\sigma_\lambda : b^\circ \to O_X$. Let $\varphi_\lambda : b^\circ \to U^\circ$ be given by $\varphi_\lambda(s) = s - \sigma_\lambda(s)$. Let $J_\lambda$ be the sheaf of two-sided ideals of $U^\circ$ generated by the image of $\varphi_\lambda$. Finally, set $D_{X, \lambda} := U^\circ / J_\lambda$.

**Theorem 1.16** $D_{X, \lambda}$ is an HTDO. Moreover, the map $\lambda \mapsto D_{X, \lambda}$ is an isomorphism between $I(b^\circ_0)$ and the set of isoclasses of HTDO on $X$.

For a proof of Theorem 1.16 see, for example, [ Section 1.2].
1.5 $D_X$-modules.

By $D_X$-module, we mean a left module over the sheaf $D_X$ of differential operators. Clearly, every $D_X$-module is also an $O_X$-module. On the other hand, given an $O_X$-module $\mathcal{M}$, giving a $D_X$-module structure to $\mathcal{M}$ is equivalent to giving a $\mathbb{C}$-linear morphism $\nabla : \mathcal{V}_X \to \mathcal{E}nd_{\mathcal{O}}(\mathcal{M}), \theta \mapsto \nabla_\theta$ satisfying the following conditions:

1. $\nabla_{f\theta}(s) = f\nabla_\theta(s)$.
2. $\nabla_\theta(f s) = \theta(f) s + f \nabla_\theta(s)$.
3. $\nabla_{[\theta_1, \theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s)$.

For $f \in \mathcal{O}_X$, $s \in \mathcal{M}$, $\theta, \theta_1, \theta_2 \in \mathcal{X}_X$.

Note that if $\mathcal{V}$ is a vector bundle, then $\nabla$ defines a connection on $\mathcal{V}$, and condition (3) amounts to saying that this connection is flat.

A $D_X$-module is called quasi-coherent if it is quasi-coherent as an $O_X$-module. Denote by $\text{Mod}_{qc}(D_X)$ the category of quasi-coherent $D_X$-modules.

An algebraic variety $X$ is said to be $D_X$-affine if the global sections functor $\Gamma : \text{Mod}_{qc}(D_X) \to \text{Mod}(\Gamma(X, D_X))$, $\mathcal{M} \mapsto \Gamma(X, \mathcal{M})$ is exact and if every module in $\text{Mod}_{qc}(D_X)$ is generated by its global sections. Clearly, every affine algebraic variety is $D_X$-affine. We’ll see later that, for a semisimple algebraic group $G$ and a Borel subgroup $B$, the corresponding flag variety $G/B$ is $D_{G/B}$-affine.

2 $D$-modules on $G/B$.

2.1 Universal enveloping algebras and the Harish-Chandra isomorphism.

Let $\mathfrak{g}$ be a semisimple Lie algebra, and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Let $\Phi$ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$, and $\Phi^+$ a choice of positive roots. Also, let $\{\alpha_1, \ldots, \alpha_l\} \subseteq \Phi^+$ be a choice of simple roots. For each $i = 1, \ldots, l$, let $\alpha_i^\vee \in \mathfrak{h}^*$ be the coroot of $\alpha_i$, and let $\pi_i \in \mathfrak{h}^*$ be the fundamental weight dual to the coroot $\alpha_i^\vee$, that is, $\langle \pi_i, \alpha_i^\vee \rangle = \delta_{ij}$. Denote by $Q = \mathbb{Z}\Phi$ the root lattice, $Q^+$ its positive part, and $P = \sum \mathbb{Z}\pi_i$ the weight lattice. Finally, let $\rho = \frac{1}{h} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^l \pi_i$ be the Weyl vector.

In this subsection, we want to study the center $\mathfrak{z} = Z(\mathfrak{u}(\mathfrak{g}))$ of the universal enveloping algebra of $\mathfrak{g}$. Since, in particular, $\mathfrak{z}$ commutes with $\mathfrak{h}$, any element $z \in \mathfrak{z}$ acts as a scalar on any highest weight module $M = \mathfrak{u}(\mathfrak{g})\psi_\lambda$ with highest weight $\lambda \in \mathfrak{h}^*$. Since every such module is a quotient of the Verma module $\Delta_\lambda$, this scalar only depends on $\lambda$. Then, for any $z \in \mathfrak{z}$, we get a function $\Xi_z : \mathfrak{h}^* \to \mathbb{C}$.

We show that $\Xi_z$ is polynomial. Indeed, it follows by the PBW theorem that $\mathfrak{u}(\mathfrak{g}) = \mathfrak{u}(\mathfrak{h}) \bigoplus (n_- \mathfrak{u}(\mathfrak{g}) + \mathfrak{u}(\mathfrak{g}) n_+)$. Since $\mathfrak{h}$ is abelian, $\mathfrak{u}(\mathfrak{h}) = S\mathfrak{h}$, the symmetric algebra. Consider then the projection $pr : \mathfrak{u}(\mathfrak{g}) \to \mathfrak{u}(\mathfrak{h})$. It follows that, for any highest weight module $M = \mathfrak{u}(\mathfrak{g})\psi_\lambda$, and any $u \in \mathfrak{u}(\mathfrak{g})$, $wu = pr(u(\lambda)v_\lambda + \text{terms of lower weight}$. Then, $\Xi_z(\lambda) = pr(z)(\lambda)$, so $\Xi_z$ is indeed polynomial. We get a map $\Xi : \mathfrak{z} \to \mathfrak{u}(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]$, which is clearly an algebra morphism.

Remark 2.1 Even though the restriction of $pr : \mathfrak{u}(\mathfrak{g}) \to \mathfrak{u}(\mathfrak{h})$ to $\mathfrak{z}$ is an algebra homomorphism, the map $pr$ is not.

Now consider the automorphism $t_{-\rho} : \mathbb{C}[\mathfrak{h}^*], t_{-\rho} : f(\lambda) \mapsto f(\lambda - \rho)$, and let $HC : \mathfrak{z} \to S(\mathfrak{h})$ be $HC = t_{-\rho} \Xi$, so that any $z \in \mathfrak{z}$ acts on any highest weight representation with highest weight $\lambda$ by $HC(z)(\lambda + \rho)$. The reason for twisting the morphism $\Xi$ is that $HC$ has its image in $S(\mathfrak{h})^W$. Indeed, it is known that, for any integral dominant weight $\lambda$ and any $w \in W$, there exists a nonzero morphism $\Delta_{w^* \lambda} \to \Delta_\lambda$, where $w \ast \lambda = w(\lambda + \rho) - \rho$ is called the $\rho$-shifted action of $W$ on $\mathfrak{h}$. It follows that $\Xi(z)(\lambda) = \Xi(z)(w \ast \lambda)$, or, equivalently, that $HC(z)(\lambda) = HC(z)(w \ast \lambda)$. Since the lattice of integral dominant weights is dense in Zariski topology, it follows that $HC(z) \in S(\mathfrak{h})^W$.

Theorem 2.2 The morphism $HC : \mathfrak{z} \to S(\mathfrak{h})^W$ is an isomorphism. It is called the Harish-Chandra isomorphism.

A strategy to prove Theorem 2.2 is to compare $HC$ with the Chevalley isomorphism. Recall that this is an isomorphism $\text{res} : S(\mathfrak{g})^G \to S(\mathfrak{h})^W$ that is given by restriction of a polynomial map in $\mathfrak{g}^*$ to $\mathfrak{h}^*$. On the other hand, we have an isomorphism of $\mathfrak{g}$-modules $\text{sym} : S(\mathfrak{g}) \to \mathfrak{u}(\mathfrak{g})$, given by
Theorem 2.5 (Borel-Weil-Bott) Consider the trivial vector bundle \( B \) at \( B \) lie in the same \( W \) group. Recall that, since \( B \) is a maximal torus with \( B \) of \( B \) precisely \( U \). In particular, \( G \) is generated by its global sections. This descends to a \( HC \) on \( G \). This diagram is not commutative. However, it is commutative ‘up to lower degree terms’, as follows. Recall that each \( \lambda \) is antidominant, then the line bundle \( L \) is ample if and only if \( \lambda \) is antidominant and regular.

\[
\text{sym} : x_1 \ldots x_n \mapsto \frac{1}{n!} \sum_{s \in S_n} x_{s(1)} x_{s(2)} \ldots x_{s(n)}.
\]

This map induces an isomorphism of vector spaces \( \text{sym} : S(\mathfrak{g})^G \to \mathfrak{z} \) whenever \( G \) is connected.

We have the following diagram,

\[
\begin{array}{ccc}
S(\mathfrak{g})^G & \xrightarrow{\text{sym}} & \mathfrak{z} \\
\downarrow{HC} & \xrightarrow{\text{res}} & S(\mathfrak{h})^W \\
\end{array}
\]

This diagram is not commutative. However, it is commutative ‘up to lower degree terms’, as follows. Recall that each \( \lambda \) is antidominant, then the line bundle \( L \) is ample if and only if \( \lambda \) is antidominant and regular.

Lemma 2.3 For any \( p \in S^n(\mathfrak{g}) \), \( \text{pr}(\text{sym}(p)) \equiv \text{res}(p) \mod S^{n-1}(\mathfrak{h}) \).

Proof. Let \( p = (\prod_{\alpha \in \Phi^+} f_{\alpha}^{k_{\alpha}})(\prod_i h_i^{n_i})(\prod_{\alpha \in \Phi^+} e_{\alpha}^{m_{\alpha}}) \in S(\mathfrak{g})^n \). Then,

\[
\text{pr}(\text{sym}(p)) = \begin{cases} 
\prod_i h_i^{n_i}, & \text{if } m_{\alpha}, k_{\alpha} = 0, \\
0, & \text{otherwise}
\end{cases} \mod S^{n-1}(\mathfrak{h}).
\]

The same is true for \( \text{res}(p) \). Then, the result follows. \( \square \)

An algebra homomorphism \( \mathfrak{z} \to \mathbb{C} \) is called a central character. For any \( \lambda \in \mathfrak{h}^* \), define a central character \( \chi_\lambda : \mathfrak{z} \to \mathbb{C} \) by defining \( \chi_\lambda(z) = HC(z)(\lambda) \). Theorem 2.2 has the following easy consequence.

Corollary 2.4 Any central character is of the form \( \chi_\lambda \) for some \( \lambda \in \mathfrak{h}^* \). Moreover, \( \chi_\lambda = \chi_\mu \) if and only if \( \lambda \) and \( \mu \) lie in the same \( W \)-orbit.

2.2 Equivariant vector bundles on the flag variety.

Let \( T \subseteq B \) be a maximal torus with \( \text{Lie}(T) = \mathfrak{h} \), and \( N \) be the unipotent radical of \( B \) so that \( B = NT \). Note that the fiber at \( B \in B \) of any \( G \)-equivariant vector bundle is a representation of \( B \). Conversely, given a representation \( U \) of \( B \), consider the trivial vector bundle \( U \times G \) on \( G \). This descends to a \( G \)-equivariant vector bundle on \( B \) whose fiber at \( B \) is precisely \( U \). In other words, \( G \)-equivariant vector bundles of \( B \) are in one-to-one correspondence with representations of \( B \). In particular, \( G \)-equivariant line bundles on \( B \) are in one-to-one correspondence with 1-dimensional representations of \( B \). These, in turn, are in one-to-one correspondence with characters of \( T \), as \( N \) acts trivially on any 1-dimensional representation of \( B \). Then, for any character \( \lambda \in P = \text{Hom}(T, \mathbb{C}^\times) \subseteq \mathfrak{h}^* \), we get a \( G \)-equivariant line bundle \( L(\lambda) \). Note that, since \( L(\lambda) \) is \( G \)-equivariant, the sheaf of differential operators \( \mathcal{D}_{\mathcal{O}_S}^{L(\lambda)} \) is an HTDO.

Theorem 2.5 (Borel-Weil-Bott) Let \( \lambda \in P \). Then, we have,

1. If \( \lambda \) is antidominant, then the line bundle \( L(\lambda) \) is generated by its global sections.
2. The line bundle \( L(\lambda) \) is ample if and only if \( \lambda \) is antidominant and regular.
3. If \( \lambda - \rho \) is not regular, then \( H^i(B, L(\lambda)) = 0 \) for \( i \geq 0 \).
4. If \( \lambda - \rho \) is regular, then there exists \( w \in W \) such that \( w \cdot \lambda := w(\lambda - \rho) + \rho \) is antidominant, and

\[
H^i(X, L(\lambda)) = \begin{cases} 
L^{-}(w \cdot \lambda) & \text{if } i = l(w), \\
0 & \text{otherwise}.
\end{cases}
\]

Where \( L^{-}(w \cdot \lambda) \) is the irreducible module with lowest weight \( w \cdot \lambda \).

Note that it follows that if \( \lambda \) is antidominant, then \( \Gamma(X, L(\lambda)) = L^{-}(\lambda) \), and \( H^i(X, L(\lambda)) = 0 \) for \( i > 0 \).
2.3 HTDO on the flag variety.

Let $G$ be a semisimple algebraic group with Lie algebra $\mathfrak{g}$, and let $B$ be a Borel subgroup, with $\mathcal{B} = G/B$ the corresponding flag variety. Then, $\mathcal{B}$ is an homogeneous $G$-variety, so we can apply the results of Subsection 1.4 to $\mathcal{B}$.

Recall the sheaf $\mathfrak{g}^o = \mathcal{O}_B \otimes _{\mathbb{C}} \mathfrak{g}$, which is the sheaf of sections of the trivial bundle $\mathcal{B} \times \mathfrak{g}$. Inside this bundle, we have the homogeneous bundle of Borel subalgebras, $\mathcal{F}$, whose fiber at a point $x \in \mathcal{B}$ is the Borel subalgebra $\mathfrak{b}_x$ corresponding to $x$. Let $\mathfrak{b}^o$ be the sheaf of sections of this bundle. Also, we have the homogeneous bundle whose fiber at each point $x$ is $[\mathfrak{b}_x, \mathfrak{b}_x]$. Let $\mathfrak{n}^o$ be the sheaf of sections of this bundle. Clearly, we have $\mathfrak{n}^o \hookrightarrow \mathfrak{b}^o \hookrightarrow \mathfrak{g}^o$.

Recall the epimorphism $r^o : \mathfrak{g}^o \to \mathcal{V}_B$. Its kernel is precisely $\mathfrak{b}^o$. Pick a basepoint $x_0 = B \in \mathcal{B}$. Then, we have $\mathfrak{b}_0 = \mathfrak{b}$. An element $\lambda \in \mathfrak{h}^*$ determines a $B$-invariant function $\mathfrak{b} \to \mathbb{C}$, and therefore a $G$-equivariant map $\lambda^o : \mathfrak{b}^o \to \mathcal{O}_B$. Let $\mathcal{J}_\lambda$ be the ideal in $\mathcal{U}^\bullet$ generated by elements of the form $\xi - (\lambda + \rho)^\gamma(\xi)$ for $\xi \in \mathfrak{b}^o$, and let $\mathcal{D}_\lambda := \mathcal{U}^\bullet / \mathcal{J}_\lambda$. The following is then a consequence of Theorem 1.16.

**Proposition 2.6** $\mathcal{D}_\lambda$ is an HTDO.

We remark that $\mathcal{D}_{-\rho} = \mathcal{D}_B$, and that for an integral weight $\lambda$, $\mathcal{D}_\lambda$ is the sheaf of differential operators on the line bundle $\mathcal{L}(\lambda + \rho)$.

By definition of an HTDO, we have a $G$-equivariant algebra homomorphism $\Psi_\lambda : \mathcal{U}(\mathfrak{g}) \to \Gamma(\mathcal{B}, \mathcal{D}_\lambda)$. Recall the central character $\chi_\lambda : \mathfrak{z} \to \mathbb{C}$. Let $\mathcal{J}_\lambda$ be the ideal of $\mathcal{U}(\mathfrak{g})$ generated by $\text{Ker}(\chi_\lambda) \subseteq \mathfrak{z}$. Let $\mathcal{U}_\lambda := \mathcal{U}(\mathfrak{g}) / \mathcal{J}_\lambda$.

**Lemma 2.7** For any $\lambda \in \mathfrak{h}^*$, the morphism $\Psi_\lambda : \mathcal{U}(\mathfrak{g}) \to \Gamma(\mathcal{B}, \mathcal{D}_\lambda)$ factors through $\mathcal{U}_\lambda$.

**Proof.** Since the map $\Psi_\lambda : \mathcal{U}(\mathfrak{g}) \to \Gamma(\mathcal{B}, \mathcal{D}_\lambda)$ is $G$-equivariant, it is enough to show that the induced map on fibers maps $\text{Ker}(\chi_\lambda)$ to 0. Note that the fibers of $\mathcal{D}_\lambda$ have the form $\mathcal{U}(\mathfrak{g}) / \sum_{x \in \mathfrak{b}} (x - (\lambda + \rho, x)) \mathcal{U}(\mathfrak{g})$. Then, for $z \in \mathfrak{z}$, by the PBW theorem, $z \in \mathfrak{n} U(\mathfrak{g}) + f$ for a unique $f \in U(\mathfrak{h})$, so that $\chi_\lambda(z) = f(\lambda + \rho)$. It follows that if $f(\lambda + \rho) = 0$, then $z = \sum_{x \in \mathfrak{b}} (x - (\lambda + \rho, x)) \mathcal{U}(\mathfrak{g})$. $\square$

Abusing notation, we denote by $\Psi_\lambda$ the induced morphism $\Psi_\lambda : \mathcal{U}_\lambda \to \Gamma(\mathcal{B}, \mathcal{D}_\lambda)$.

**Theorem 2.8 (Beilinson-Bernstein)** For any $\lambda \in \mathfrak{h}^*$, the morphism $\Psi_\lambda : \mathcal{U}_\lambda \to \Gamma(\mathcal{B}, \mathcal{D}_\lambda)$ is an isomorphism.

The key step to prove Theorem 2.8 is to prove its quasiclassical version, that is, to prove that its associated graded morphism is an isomorphism. In fact, we are going to relate this isomorphism to the Springer resolution.

By the PBW theorem, we know that $\text{gr} \mathcal{U}(\mathfrak{g}) = S(\mathfrak{g})$. We canonically identify $\mathfrak{g} \cong \mathfrak{g}^*$ via the Killing form, so $\text{gr} \mathcal{U}(\mathfrak{g}) \cong \mathcal{C}[\mathfrak{g}]$. The next Lemma characterizes the ideal of the nilpotent cone $\mathcal{N}$ in $S(\mathfrak{g})$.

**Lemma 2.9** The ideal in $S(\mathfrak{g})$ defining $\mathcal{N}$ is generated by $S(\mathfrak{g})^G_+ = S(\mathfrak{g})^G \cap (\bigoplus_{p > 0} S(\mathfrak{g})_p)$.

**Proof.** Recall the Chevalley isomorphism, $\text{res} : S(\mathfrak{g})^G \to S(\mathfrak{h})^W$. This maps the ideal $S(\mathfrak{g})^G_+$ to $S(\mathfrak{h})^W$, which is the ideal of 0 in $\mathfrak{h} / W$. Then, the ideal generated by $S(\mathfrak{g})^G_+$ defines the elements $x \in \mathfrak{g}^*$ such that $G x \cap \mathfrak{h} = 0$, that is, elements whose Jordan decomposition doesn’t have a semisimple part. This is precisely $\mathcal{N}$. By results of Kostant, $\mathcal{N}$ is a normal variety and a complete intersection in $\mathfrak{g}$. The result now follows. $\square$

Note that, by lifting elements of $S(\mathfrak{g})^G_+$ to $\mathcal{U}(\mathfrak{g})$, it follows that we have an epimorphism $\mathcal{C}[\mathcal{N}] \to \text{gr} \mathcal{U}_\lambda$. We then have the following commutative diagram:

\[
\begin{array}{ccc}
S(\mathfrak{g}) & \longrightarrow & \text{gr} \Gamma(\mathcal{B}, \mathcal{D}_\lambda) \\
\downarrow & & \downarrow \\
\text{gr} \mathcal{U}_\lambda & \downarrow & \\
& \downarrow & \\
& \mathcal{C}[\mathcal{N}] & \\
\end{array}
\]

Recall the Springer resolution $\gamma : T^* \mathcal{B} \to \mathcal{N}$, and its pullback $\gamma^* : \mathcal{C}[\mathcal{N}] \to \Gamma(T^* \mathcal{B}, \mathcal{O}_{T^* \mathcal{B}})$. Since $\gamma$ is a resolution of singularities and $\mathcal{N}$ is normal, $\gamma^*$ is actually an isomorphism. Moreover, the following diagram commutes.
Since \( \Gamma(B, D_\lambda) \to \Gamma(B, g_\lambda D_\lambda) \) is injective, it follows that \( \mathbb{C}[N] \to g_\lambda U_\lambda \to \Gamma(B, D_\lambda) \) and \( \Gamma(B, g_\lambda D_\lambda) \to \Gamma(B, g_\lambda U_\lambda) \) are all isomorphisms. Then, \( g_\lambda \Psi_\lambda \) is an isomorphism. Since all the algebras we’re working with are positively graded, it follows that \( \Psi_\lambda \) is an isomorphism.

We have two induced functors. The first functor is the \textit{global sections functor}:

\[
\text{Mod}_{qc}(D_\lambda) \to \text{Mod}_{-}(U(g))
\]

\[
\mathcal{M} \mapsto \Gamma(B, \mathcal{M}).
\]

Note that \( \Gamma(B, \mathcal{M}) = \text{Hom}_{O_B}(O_B, \mathcal{M}) \). Also, note that \( \text{Hom}_{O_B}(O_B, \mathcal{M}) = \text{Hom}_{D_\lambda}(D_\lambda, \mathcal{M}) \): if \( \mathcal{M} \) has a \( D_\lambda \)-module structure, then any \( O_B \)-homomorphism \( O_B \to \mathcal{M} \) admits a unique extension to a \( D_\lambda \)-linear homomorphism \( D_\lambda \to \mathcal{M} \).

The next functor is the \textit{localization functor}:

\[
\text{Mod}_{-}(U(g)) \to \text{Mod}_{qc}(D_\lambda)
\]

\[
\mathcal{M} \mapsto D_\lambda \otimes_{U(g)} \mathcal{M}.
\]

Note that, even if \( U(g) \) is an algebra and \( D_\lambda \) a sheaf, the localization functor makes sense. Indeed, it follows by Theorem 2.8 that, for every open set \( U \), \( D_\lambda(U) \) is an \( U(g) \) module. Note that the global sections functor is right adjoint to the localization functor. Our next goal is to study these functors.

### 2.4 Abelian Beilinson-Bernstein theorem.

The goal of this subsection is to state and prove two fundamental theorems of Beilinson-Bernstein on the cohomology of \( O_B \)-coherent \( D_\lambda \)-modules. The first one, Theorem 2.11, concerns the vanishing of the higher cohomology of modules. The second Theorem 2.15 tells us when every \( O_B \)-coherent \( D_\lambda \)-module is generated by its global sections. A strategy to do this is to realize every \( O_B \)-coherent submodule of such a module as a direct summand in a sheaf without higher cohomology. To do this, we will use the Borel-Weil-Bott theorem, which tells us that the sheaf \( L(\lambda) \) is ample whenever \( \lambda \in P \) is antidominant and regular.

Assume that \( \mu \in P \) is antidominant. By the Borel-Weil-Bott theorem, Theorem 2.5 (1), \( L(\mu) \) is generated by its global sections. We know that the global sections of \( L(\mu) \) are \( L(\mu) \), the simple module with lowest weight \( \mu \). Then, we have \( p_\mu : O_B \otimes_{\mathbb{C}} L(\mu) \to L(\mu) \). Taking the dual of \( p_\mu \), we get a morphism \( \text{Hom}_{O_B}(L(\mu), O_B) \to O_B \otimes_{\mathbb{C}} \text{Hom}_{\mathbb{C}}(L^-(\mu), \mathbb{C}) \). This is injective. Rewriting, we have an injective morphism \( L(\mu) \to O_B \otimes_{\mathbb{C}} L^+(\mu) \). If we tensor with the locally free module \( L(\mu) \), we get an injective morphism,

\[
i_\mu : O_B \to L(\mu) \otimes_{\mathbb{C}} L^+(\mu).
\]

Tensoring with an \( O_B \)-coherent \( D_\lambda \) module \( \mathcal{M} \) we get,

\[
i_{\mu,\mathcal{M}} : \mathcal{M} \to \mathcal{M} \otimes_{O_B} L(\mu) \otimes_{\mathbb{C}} L^+(\mu).
\]

Note that \( i_{\mu,\mathcal{M}} \) is always injective because \( i_\mu \) locally splits. We want to show that, if \( \lambda \) is antidominant, \( i_{\mu,\mathcal{M}} \) splits as morphism of sheaves of vector spaces. Note that this splitting will be constructed using differential operators, so it is not a splitting of \( O_B \)-modules. For example, take \( g = \mathfrak{sl}_2 \), so \( B = \mathbb{P}^1 \), and \( \mu = -1 \). To produce a splitting of \( i_{\mu,\mathcal{M}} \) as sheaf of vector spaces, we will realize the image of \( i_{\mu,\mathcal{M}} \) as a generalized eigensheaf for the action of the center \( \mathfrak{z} \) of \( U(g) \) on \( \mathcal{M} \otimes_{O_B} L(\mu) \otimes_{\mathbb{C}} L^+(\mu) \).

An essential ingredient will be the following construction. Let \( F \) be a finite dimensional \( g \)-module. Recall that \( F \) has a filtration by \( b \)-submodules \( 0 = F_0 \subset F_1 \subset \cdots \subset F_m = F \), where \( \dim F_i = i \), \( n F_i \subset F_{i-1} \) and \( \mathfrak{h} \) acts on the \( 1 \)-dimensional quotient \( F_i/F_{i-1} \) by an integral character \( \nu_i \). The \( \nu_i \)'s are just the weights of \( F \). Consider the trivial vector
bundle $B \times F \to B$. Its sheaf of sections is $\mathcal{F} := \mathcal{O}_B \otimes_C F$. Note that $B \times F$ has a filtration $0 = U_0 \subset U_1 \subset \cdots \subset U_m$, where

$$U_i := \{(gB, v) \in B \times F : v \in g(F_i)\}.$$

This defines a filtration on $F$, $0 = F_0 \subset F_1 \subset \cdots \subset F_m = F$. These are $G$-equivariant coherent sheaves on $B$. Recall that we have an equivalence between the category of $G$-equivariant coherent sheaves on $B$ and the category of representations of the Borel subgroup $B$. Under this equivalence, $F_i$ corresponds to $F_i$. It then follows that $F_i / F_{i-1} = L(\nu_i)$.

It follows that, more generally, for any quasi-coherent $O_B$-module $M$, $M \otimes_C F$ has a filtration with successive quotients being $M \otimes_{O_B} L(\nu_i)$. Now assume that $M$ is a $D_\lambda$-module. Then, $M$ is a $g^\circ = O_B \otimes_C g$-module such that the subbundle of Borel subalgebras $b^\circ$ acts with character $(\lambda + \rho)^\circ$. Similarly, $b^\circ$ acts on $L(\nu_i)$. It follows that $M \otimes_{O_B} L(\nu_i)$ is a $g^\circ$-module and the $b^\circ$ acts on it with character $(\lambda + \nu_i + \rho)^\circ$. In other words, the action of $U^\circ$ on $M \otimes_{O_B} L(\nu_i)$ factors through the quotient $D_{\lambda+\nu_i}$. By Theorem 2.8 the center $\mathfrak{g}$ of $U(\mathfrak{g})$ acts on $M \otimes_{O_B} L(\nu_i)$ with character $\chi_{\lambda+\nu_i}$. It follows that

$$\prod_i (z - \chi_{\lambda+\nu_i}(z))$$

annihilates $M \otimes_C F$ for every $z \in \mathfrak{g}$. Then, the action of $\mathfrak{g}$ on $M \otimes_C F$ is locally finite, and $M \otimes_C F$ decomposes into the direct sum of its generalized $\mathfrak{g}$-eigensheaves.

For a $U^\circ$-module $M$ and $\lambda \in \mathfrak{h}^*$, denote by $M_{[\lambda]}$ the generalized $\mathfrak{g}$-eigensheaf of $M$ with eigencharacter $\chi_\lambda$. Note that $M_{[\lambda]} = [M] = [\mu]$ whenever $\lambda, \mu$ belong to the same $W$-orbit.

**Lemma 2.10** Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then, for every $O_B$-quasi-coherent $D_\lambda$-module $M$, and every antidominant integral weight $\mu$, $i_{\mu, M}$ splits. In particular, $M \cong [M \otimes_{O_B} L(\mu) \otimes_C L^+(-\mu)]_{[\lambda]}$.

**Proof.** We know that the eigencharacters of $M \otimes_{O_B} L(\mu) \otimes_C L^+(-\mu)$ are of the form $\chi_{\lambda+\mu+\nu_i}$ where $\nu_i$ is a weight of $L^+(-\mu)$. Assume that $\chi_{\lambda+\mu+\nu_i} = \chi_\lambda$ for some weight $\nu_i$ of $L^+(-\mu)$. Then, for some $w \in W$, $w(\lambda) = \lambda + \mu + \nu_i$, so $(-\mu - \nu_i) + w(\lambda) = 0$. But $\lambda$ is antidominant, so $w(\lambda) - \lambda$ is positive, that is, it is a non-negative linear combination of simple roots. Since $L^+(-\mu)$ is the irreducible module with highest weight $-\mu$, $-\mu - \nu_i$ is also positive. It follows that $w(\lambda) = \lambda$ and $\mu = -\nu_i$. Then, the generalized eigensheaf with eigencharacter $\chi_\lambda$ is $M \otimes_{O_B} L(\mu) \otimes_{O_B} L(-\mu) = M$.

**Theorem 2.11 (Beilinson-Bernstein)** Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then, $H^i(B, M) = 0$ for every quasi-coherent $D_\lambda$-module $M$ and $i > 0$. In particular, the global sections functor $\Gamma(B, \bullet) : \text{Mod}_{g^\circ}(D_\lambda) \to \text{Mod}_{U^\circ}$ is exact.

**Proof.** Let $W$ be an $O_B$-coherent submodule of $M$. By the Borel-Weil-Bott Theorem (Theorem 2.5(2)) we can find an antidominant weight $\mu$ such that $H^i(B, W \otimes_{O_B} L(\mu)) = 0$ for $i > 0$. Then, $H^i(B, W \otimes_{O_B} L(\mu) \otimes_C L^+(-\mu)) = 0$. Now consider the following commutative diagram:

$$
\begin{array}{ccc}
H^i(B, W) & \longrightarrow & H^i(B, M) \\
\downarrow & & \downarrow \\
0 = H^i(B, W \otimes_{O_B} L(\mu) \otimes_C L^+(-\mu)) & \longrightarrow & H^i(B, M \otimes_{O_B} L(\mu) \otimes_C L^+(-\mu))
\end{array}
$$

Since the diagram commutes and, by the previous lemma, $H^i(B, M) \to H^i(B, M \otimes_{O_B} L(\mu) \otimes_C L^+(-\mu))$ is injective, we get that $H^i(B, W) \to H^i(B, M)$ is the zero map. Since $M$ is the direct limit of its $O_B$-coherent submodules and cohomology commutes with direct limits, we get that $H^i(B, M) = 0$.

**Corollary 2.12** Let $\lambda \in \mathfrak{h}^*$ be antidominant. Then, for every $U_\lambda$-module $V$, the natural map $\varphi_V$ from $V$ to $\Gamma(B, O_B \otimes_{U(\mathfrak{g})} V)$ is an isomorphism of $g$-modules.

**Proof.** By the previous theorem, the global sections functor $\Gamma$ is exact. Then, the functor $\Gamma(B, O_B \otimes_{U(\mathfrak{g})})$ is right exact. Now let $V \in \text{Mod}_{U_\lambda}$. There exists an exact sequence $(U_\lambda)^{\oplus i} \to (U_\lambda)^{\oplus j} \to V \to 0$. Then, we get a commutative diagram,

---

Note that the argument I gave on the October 18 talk is incorrect: the morphism $i_{\mu, W}$ does not necessarily split.
Proof. Follows from Corollary 2.13 and Theorem 2.15.

\[ \text{Mod} \]

Corollary 2.16

Let \( D \) be an antidominant algebra whose associated graded algebra has finite homological dimension. Moreover, the homological dimension \( \lambda \) hypothesis, Proposition 2.17

If \( \lambda \) is regular. It is known that if this is not the case then the homological dimension of \( D \) is finite whenever \( \lambda \) is antidominant and regular. Theorem 2.11 but somewhat easier. Recall that for any integral antidominant weight \( \mu \), \( D \) is generated by its global sections. The strategy is similar to that of the proof of Lemma 2.10.

\[ \text{Mod} \]

\[ \text{Mod} \]

\[ \text{Mod} \]

\[ \text{Mod} \]

The first two vertical maps are isomorphisms. Then, the third vertical map is also an isomorphism. \( \square \)

Corollary 2.13

Let \( \lambda \in \mathfrak{h}^* \) be antidominant. Then, the localization functor induces an equivalence of \( \text{Mod} \mathcal{U}_\lambda \) to \( \text{Qmod}_{qc}(D_\lambda) \).

Proof. Let \( \mathcal{M} \in \text{Qmod}_{qc}(D_\lambda) \). By adjointness, we have a natural morphism \( \psi_\mathcal{M}: D_\lambda \otimes \mathcal{U}_\lambda \Gamma(B, \mathcal{M}) \to \mathcal{M} \). Let \( \mathcal{K}' \) and \( \mathcal{K}'' \) be the kernel and cokernel of this morphism, respectively. Then, we get an exact sequence \( 0 \to \mathcal{K}' \to D_\lambda \otimes \mathcal{U}_\lambda \Gamma(B, \mathcal{M}) \to \mathcal{K}'' \to 0 \). Applying the global sections functor we find that \( \Gamma(B, \mathcal{K}') = 0 \), \( \Gamma(B, \mathcal{K}'') = 0 \). The result follows. \( \square \)

Now we show another result due to Beilinson-Bernstein, that says that when \( \lambda \in \mathfrak{h}^* \) is antidominant and regular, every quasi-coherent \( D_\lambda \)-module \( \mathcal{M} \) is generated by its global sections. The strategy is similar to that of the proof of Theorem 2.11 but somewhat easier. Recall that for any integral antidominant weight \( \mu \) we have a surjective morphism \( p_\mu: \mathcal{O}_B \otimes \mathcal{L}^-(\mu) \to \mathcal{L}(\mu) \). Note that this morphism locally splits. Then, for every quasi-coherent module \( \mathcal{M} \) we get an epimorphism \( p_{\mu, \mathcal{M}}: \mathcal{M} \otimes \mathcal{L}^-(\mu) \to \mathcal{M} \otimes \mathcal{O}_B \mathcal{L}(\mu) \). The following is an analog of Lemma 2.10.

**Lemma 2.14.** Assume that \( \lambda \) is antidominant and regular. Then, for every quasi-coherent \( D_\lambda \)-module \( \mathcal{M} \), and every antidominant integral weight \( \mu \), the epimorphism \( p_{\mu, \mathcal{M}} \) splits. In fact, the generalized \( \chi_{\lambda+\mu} \)-eigensheaf of \( \mathcal{M} \otimes \mathcal{L}^-(\mu) \) is \( \mathcal{M} \otimes \mathcal{O}_B \mathcal{L}(\mu) \).

Proof. We argue similarly to Lemma 2.10. Assume there exists a weight \( \nu_\mathcal{M} \) of \( \mathcal{L}^-(-\mu) \) and \( w \in \mathcal{W} \) such that \( w(\lambda + \nu) = \lambda + \nu \). Then, \( (w(\lambda) - \lambda) + (w(\nu) - \lambda) = 0 \). Similarly to Lemma 2.10 it follows that \( \nu \) is regular. The result follows. \( \square \)

**Theorem 2.15 (Beilinson-Bernstein).** Let \( \lambda \in \mathfrak{h}^* \) be antidominant and regular. Then, for any quasi-coherent \( D_\lambda \)-module \( \mathcal{M} \), the morphism \( D_\lambda \otimes_{\mathcal{U}(\mathfrak{b})} \Gamma(B, \mathcal{M}) \to \mathcal{M} \) is surjective. In other words, every quasi-coherent \( D_\lambda \)-module is generated by its global sections.

Proof. Since \( \lambda \) is antidominant, \( \Gamma(B, \mathcal{M}) \) is exact. Hence, it suffices to show that \( \Gamma(B, \mathcal{M}) \neq 0 \) for \( \mathcal{M} \neq 0 \). \( \square \)

We can assume that \( \mathcal{M} \) is coherent. By the Borel-Weil-Bott Theorem, we can find a regular antidominant weight \( \nu \) such that \( \Gamma(B, \mathcal{M} \otimes_{\mathcal{O}_B} \mathcal{L}(\nu)) \neq 0 \). Since \( \nu \) is regular, Lemma 2.14 implies that \( \mathcal{L}^-(\nu) \otimes \Gamma(B, \mathcal{M}) \neq 0 \). We’re done. \( \square \)

**Corollary 2.16.** Let \( \lambda \in \mathfrak{h}^* \) be antidominant and regular. Then, the global sections functor is an equivalence of categories \( \text{Qmod}_{qc}(D_\lambda) \to \text{Mod} \mathcal{U}_\lambda \). Its inverse is the localization functor.

Proof. Follows from Corollary 2.13 and Theorem 2.15. \( \square \)

As an application of Theorems 2.11, 2.15 we show that the homological dimension of \( \mathcal{U}_\lambda \) is finite whenever \( \lambda \) is regular. It is known that if this is not the case then the homological dimension of \( \mathcal{U}_\lambda \) is infinite.

**Proposition 2.17.** Let \( \lambda' \in \mathfrak{h}^* \) be regular. Then the homological dimension of \( \mathcal{U}_{\lambda'} \) is finite.

Proof. Since \( \mathcal{U}_\lambda = \mathcal{U}_{\lambda W} \) for any \( w \in \mathcal{W} \), we can replace \( \lambda' \) by an antidominant weight \( \lambda \in \mathcal{W} \lambda' \). Note that, by hypothesis, \( \lambda \) is regular. Since \( \mathcal{D}_\lambda \) is a TDO, the homological dimension of each stalk \( \mathcal{D}_{\lambda, x} \) is finite, as this is a filtered algebra whose associated graded algebra has finite homological dimension. Moreover, the homological dimension \( \hd \mathcal{D}_{\lambda, x} \leq \dim \mathcal{B} \), so that these homological dimensions are uniformly bounded. It is known that, for any \( x \in \mathcal{B}, i \in \mathbb{Z}_{>0} \), we have \( \mathcal{E}xt^i_{\mathcal{D}_\lambda}(\mathcal{M}, \mathcal{W})_x = \mathcal{E}xt^i_{\mathcal{D}_{\lambda, x}}(\mathcal{M}_x, \mathcal{W}_x) \), for an \( \mathcal{O}_B \)-coherent \( \mathcal{D}_\lambda \)-module \( \mathcal{M} \) and a quasi-coherent \( \mathcal{D}_\lambda \)-module \( \mathcal{W} \). Then, \( \mathcal{E}xt^i_{\mathcal{D}_\lambda}(\mathcal{M}, \mathcal{W}) = 0 \) for \( i > \dim \mathcal{B} \).

On the other hand, we have the Grothendieck spectral sequence \( \mathcal{H}^p(\mathcal{B}, \mathcal{E}xt^q_{\mathcal{D}_\lambda}(\mathcal{M}, \mathcal{W})) \Rightarrow \mathcal{E}xt^p_{\mathcal{D}_\lambda}(\mathcal{M}, \mathcal{W}) \). It follows that \( \mathcal{E}xt^i_{\mathcal{D}_\lambda}(\mathcal{M}, \mathcal{W}) = 0 \) for \( i > 2 \dim \mathcal{B} \), where \( \mathcal{M} \) is a coherent \( \mathcal{D}_\lambda \)-module and \( \mathcal{W} \) is a quasi-coherent \( \mathcal{D}_\lambda \)-module.
Since we’re assuming \( \lambda \) is antiderminant and regular, \( \text{Ext}_U^i(M,W) = 0 \) for any finitely generated \( U_\lambda \)-module \( M \), any \( U_\lambda \)-module \( W \) and any \( i > 2 \dim B \). Taking direct limits, it follows that \( \text{Ext}_U^i(M,W) = 0 \) for any two \( U_\lambda \)-modules \( M,W \) and any \( i > 2 \dim B \). Then, \( \text{hd} U_\lambda \leq 2 \dim B \). \( \square \)

**Remark 2.18** If \( \lambda \in \mathfrak{h}^* \) is an integral regular weight, then actually \( \text{hd} U_\lambda = 2 \dim B \). For example, if \( \lambda = -\rho \), then \( U_\lambda = D(B) \), the algebra of (global) differential operators on \( B \). By Corollary 2.16, \( \text{Ext}^*_U(\mathbb{C}, \mathbb{C}) = \text{Ext}^*_D(O,O) = H^*(B, \mathbb{C}) \), where the last equality follows because \( B \) is smooth. Since \( B \) is a smooth, irreducible, projective variety, \( H^{2 \dim B}(B, \mathbb{C}) = \mathbb{C} \).

### 2.5 Derived Beilinson-Bernstein Theorem.

Assume that \( \lambda \in \mathfrak{h}^* \) is regular. Then, \( U_\lambda \) has finite homological dimension, so the localization functor has a left derived functor \( \mathcal{D}_\lambda \otimes_{U_\lambda} \cdot : D^b(\text{Mod-}U_\lambda) \to D^b(\text{Mod}_{qc}(\mathcal{D}_\lambda)) \). Note that the global sections functor admits a right derived functor \( R\Gamma : D^b(\text{Mod}_{qc}(\mathcal{D}_\lambda)) \to D^b(\text{Mod-}U_\lambda) \).

**Theorem 2.19** Let \( \lambda \in \mathfrak{h}^* \) be a regular integral weight. Then, \( \mathcal{D}_\lambda \otimes_{U_\lambda} \cdot \) and \( R\Gamma \) are quasi-inverse equivalences of triangulated categories.

**Remark 2.20** We remark that Theorem 2.19 is valid in a greater generality for \( \lambda \in \mathfrak{h}^* \) regular but not necessarily integral.

The rest of this section is devoted to the proof of Theorem 2.19. Let \( P \) be a projective \( U_\lambda \)-module. Recall that this means that \( P \) is a direct summand of a free module \( U_\lambda^{[i]} \) for some set \( I \) so, in particular, \( P \) is flat. Note that, by Theorem 2.8, the adjunction morphism \( P \to \Gamma(B, \mathcal{D}_\lambda \otimes_{U_\lambda} P) \) is an isomorphism. Also, it is clear that \( \mathcal{D}_\lambda \otimes_{U_\lambda} P \) is a direct summand of \( \mathcal{D}_\lambda^{[i]} \). Note that \( \mathcal{O}_{T^*B} \) has no higher cohomology: since \( T^*B \) is symplectic, the canonical bundle on \( T^*B \) is trivial, and then the vanishing of cohomology is a consequence of the Grauert-Riemenschneider Vanishing Theorem applied to \( T^*B \to N \). Since \( \text{gr} \mathcal{D}_\lambda = \mathcal{O}_{T^*B} \), it follows that \( \mathcal{D}_\lambda \) is \( \Gamma \)-acyclic. Hence, \( \mathcal{D}_\lambda \otimes_{U_\lambda} P \) is \( \Gamma \)-acyclic.

Now, let \( V \) be a complex in \( D^b(\text{Mod-}U_\lambda) \), with \( \lambda \in \mathfrak{h}^* \) a regular weight. By Proposition 2.17 \( V \) is quasi-isomorphic to a complex \( P \) of projective \( U_\lambda \)-modules, and \( \mathcal{D}_\lambda \otimes_{U_\lambda} V = \mathcal{D}_\lambda \otimes_{U_\lambda} P \). It follows that \( R\Gamma(\mathcal{D}_\lambda \otimes_{U_\lambda} V) = \Gamma(\mathcal{D}_\lambda \otimes_{U_\lambda} P) \cong P \cong V \). We have proved the following.

**Lemma 2.21** Let \( \lambda \in \mathfrak{h}^* \) be a regular weight. Then, \( R\Gamma(\mathcal{D}_\lambda \otimes_{U_\lambda} \cdot) : D^b(\text{Mod-}U_\lambda) \to D^b(\text{Mod-}U_\lambda) \) is isomorphic to the identity functor.

Note that it follows that \( R\Gamma \) is a quotient functor of triangulated categories. Then, to prove Theorem 2.19 it suffices to show that \( R\Gamma(M \cdot) = 0 \) only when \( M \cdot = 0 \). We will follow a strategy that appears in [2, Section 3]. We will need the following result, due to Kontsevich (see e.g. [2, Theorem 3.5.1]):

**Lemma 2.22** Let \( X \subseteq \mathbb{P}^n_\mathbb{C} \) be a smooth closed subscheme. Then, \( \mathcal{O}_X(i), -n \leq i \leq 0 \) generate \( D^b(\text{coh} X) \) under shifts, cones, and direct summands.

**Corollary 2.23** There exists a finite set of dominant weights \( S \) such that \( L(\mu), \mu \in S \), generate \( D^b(\text{coh} B) \) under shifts, cones, and direct summands.

We will also need a derived version of the splitting method used in the proof of Theorems 2.11, 2.13. Recall that, if \( M \) is a (sheaf of) module(s) on which the center \( \mathfrak{z} \) of \( U_\mathfrak{g} \) acts locally finitely, then by \( [M]_{\lambda} \) we denote the generalized eigenspace (resp. generalized eigensheaf) with generalized eigencharacter \( \chi_{\lambda} \). For integral weights \( \lambda, \mu \) with \( \mu - \lambda \) dominant, define the translation functor \( T^b_\lambda : \text{Mod-}U_\lambda \to \text{Mod-}U_\mu \) by \( T^b_\lambda(M) = [L^+(\mu - \lambda) \otimes M]_{\mu} \), where \( L^+(\mu - \lambda) \) is the simple finite dimensional module with highest weight \( \mu - \lambda \).

Now, let \( M \) be in \( D^b(\text{Mod}_{qc}(\mathcal{D}_\lambda)) \). Then, using the notation of the previous paragraph, \( L^+(\mu - \lambda) \otimes M \cdot \) is a complex of \( U^\mathfrak{g} \)-modules. Moreover, by the construction before Lemma 2.10, \( T^b_\lambda(M) \) acts locally finitely on \( L^+(\mu - \lambda) \otimes M \cdot \). So one can define \( T^b_\lambda(M \cdot) \) similarly as above. We have that translation functors commute with \( R\Gamma \), that is,

\[
T^b_\lambda[R\Gamma_\lambda(M \cdot)] \cong R\Gamma_\mu((T^b_\lambda(M \cdot))]
\]

(2)

We’ll use the following Lemma, that is parallel to Lemmas 2.10, 2.14. The proof is also similar. To get the desired combinatorial relations between the weights, it uses [3, Lemma 7.7].
Lemma 2.24 Assume that $\lambda$, $\mu$ are in the same chamber. Then, for $\mathcal{M} \in D^b(\text{Mod}_{qc}(\mathcal{D}_\lambda))$, $T^\mu_\lambda(\mathcal{M}) = L(\mu - \lambda) \otimes_{\mathcal{O}_B} \mathcal{M}$.

Finally, Theorem 2.19 follows from the next result.

Lemma 2.25 Let $\lambda$ be an integral and regular weight, and let $\mathcal{M} \in D^b(\text{Mod}_{qc}(\mathcal{D}_\lambda))$ be such that $R\Gamma(\mathcal{M}) = 0$. Then, $\mathcal{M} = 0$.

Proof. Let $\mu$ be a dominant weight such that $\lambda$, $\lambda + \mu$ are in the same chamber. It then follows from Equation (2) that $0 = T^{\lambda+\mu}_\lambda(\mathcal{M}) = R\Gamma_{\lambda+\mu}(L(\mu) \otimes_{\mathcal{O}_B} \mathcal{M})$. By Corollary 2.23 it follows that for $\lambda$ deep in its chamber, $R\Gamma(F \otimes \mathcal{M}) = 0$ for all $F \in D^b(\text{coh} \mathcal{B})$. Then, $\mathcal{M} = 0$.

The case where $\lambda$ is any integral regular weight follows again from (2) but, to pass from an integral regular weight $\lambda$ to another (integral and regular) weight deep into the chamber of $\lambda$, we need to extend the definition of translation functors to allow the case when the difference $\mu - \lambda$ is not dominant. Here, define $T^\mu_\lambda(\mathcal{M}) := [L(\mu - \lambda) \otimes M]_\mu$, where $L(\mu - \lambda)$ is a finite dimensional $\mathfrak{g}$-module with extremal weight $\mu - \lambda$. Again, we can extend this functor to $D^b(\text{Mod}_{qc}(\mathcal{D}_\lambda))$, and Equation (2) is valid. Finally, Lemma 2.24 is also valid in this more general setting, with the same proof. It follows that, for $\lambda, \mu$ in the same chamber and $\mathcal{M} \in D^b(\text{Mod}_{qc}(\mathcal{D}_\lambda))$, we have $T^\mu_\lambda(\mathcal{M}) = 0$ only when $\mathcal{M} = 0$. □

References