HARISH-CHANDRA BIMODULES OVER RATIONAL CHEREDNIK ALGEBRAS

JOSÉ SIMENTAL

Abstract. We study Harish-Chandra bimodules over the rational Cherednik algebra \( H_c(W) \) associated to a complex reflection group \( W \) with parameter \( c \). Our results allow us to partially reduce the study of these bimodules to smaller algebras. We classify those pairs of parameters \((c, c')\) for which there exist nonzero fully supported Harish-Chandra bimodules, and give a description of the category of all Harish-Chandra bimodules modulo those without full support. When \( W \) is a symmetric group we are able to classify all irreducible Harish-Chandra bimodules. Our proofs are based on localization techniques and the study of partial KZ functors.

1. Introduction

In this paper we study Harish-Chandra bimodules over rational Cherednik algebras. Recall that a rational Cherednik algebra is an associative algebra \( H_c := H_c(W, \mathfrak{h}) \) associated to a complex reflection group \( W \) and its reflection representation \( \mathfrak{h} \), see Subsection 2.1 for a precise definition. This algebra depends on a parameter \( c \), which is a conjugation invariant function \( c : \mathcal{S} \to \mathbb{C} \), where \( \mathcal{S} \) is the set of reflections of \( W \). The algebra \( H_c \) is filtered, with associated graded \( \text{gr} H_c = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]/\mathbb{C} \mathfrak{h} \), the smash-product algebra. It follows that \( H_c \) has a triangular decomposition \( H_c = \mathbb{C}[\mathfrak{h}] \otimes CW \otimes \mathbb{C}[\mathfrak{h}^*] \), where \( \mathbb{C}[\mathfrak{h}], \mathbb{C}[\mathfrak{h}^*] \) and \( CW \) sit inside \( H_c \) as subalgebras, similar to that of the universal enveloping algebra of a semisimple Lie algebra \( \mathfrak{g} \). Then, the representation theory of the rational Cherednik algebra has many similarities with that of semisimple Lie algebras. For example, one has a category \( \mathcal{O}_c \), \cite{GGOR}, to be recalled in Subsection 2.3 below, that has been extensively studied in recent years, see e.g., \cite{BE, GGOR, GL, L4, R, RSVV, Sh, Web, Wi}. One also has a notion of Harish-Chandra bimodules, \cite{BEG}. These are the main object of study of this paper. Unlike category \( \mathcal{O} \), Harish-Chandra bimodules have not been extensively studied in the literature. Let us remark that, while in the Lie algebra setting category \( \mathcal{O} \) of Harish-Chandra bimodules, \cite{BEG}. These are the main object of study of this paper. Unlike category \( \mathcal{O} \), Harish-Chandra bimodules have not been extensively studied in the literature. Let us remark that, while in the Lie algebra setting category \( \mathcal{O} \) and the category of Harish-Chandra bimodules are very similar, cf. \cite{BG}, Section 5], this is no longer the case in the Cherednik algebra setting see, for example, Subsection 6.5 below.

An \( H_c H_{c'} \)-bimodule \( B \) is said to be Harish-Chandra (HC, for short) if it is finitely generated and the adjoint action of every element from \( \mathbb{C}[\mathfrak{h}]W \) or \( \mathbb{C}[\mathfrak{h}^*]W \) is locally nilpotent, \cite{BEG, Definition 3.2]. Just as in the Lie algebra case, HC bimodules form a full Serre subcategory of the category of \( H_c H_{c'} \)-bimodules, and the tensor product with a HC \( H_c H_{c'} \)-bimodule \( B \) induces a functor \( B \otimes_{H_c} \mathcal{O}_{c'} : \mathcal{O}_{c'} \to \mathcal{O}_c \). Recently, these functors have been used to construct derived equivalences between categories \( \mathcal{O} \) for different parameters \( c, c' \), see Subsection 5.4 in \cite{L4}.

An interesting problem, then, is to describe the category \( \text{HC}(c, c') \) of HC \( H_c H_{c'} \)-bimodules. In this paper, we address the problem of classifying its irreducible objects. A classification of irreducible HC \( H_c H_{c'} \)-bimodules has been carried out in \cite{BEG} for the case when \( W \) is a Coxeter group and the parameters \( c, c' \) are integral. Namely, in this case we have that the category \( \text{HC}(c, c') \) is semisimple and isomorphic to the category of finite dimensional representations of the group \( W \). An explicit construction of its irreducibles is given in terms of spaces of locally finite maps \( \text{Hom}_{\mathcal{O}_c}(M, N) \) for irreducible objects \( M, N \in \mathcal{O}_c \). Here, a map \( f \in \text{Hom}_c(M, N) \) is said to be locally finite if it is locally nilpotent with respect to the operator \( \text{ad}(a) \) for all \( a \in \mathbb{C}[\mathfrak{h}]W \cup \mathbb{C}[\mathfrak{h}^*]W \). Moreover, for \( B_1 \in \text{HC}(c, c'), B_2 \in \text{HC}(c', c'') \) one has that \( B_1 \otimes_{H_{c'}} B_2 \in \text{HC}(c, c'') \), Proposition 2.2. So the category \( \text{HC}(c, c) \) becomes a monoidal category and the equivalence \( \text{HC}(c, c) \cong W\text{-rep} \) is that of monoidal categories.

In this paper, we generalize the previous result by giving a description of the category \( \text{HC}(c, c') \) of all HC \( H_c H_{c'} \)-bimodules modulo the full subcategory whose objects are HC bimodules whose singular support is strictly contained in \( (\mathfrak{h} \oplus \mathfrak{h}^*)/W \), see Subsection 2.8 for the definition of the singular support of a HC bimodule. Here, we allow \( W \) to be any complex reflection group, and the parameters \( c, c' \) are arbitrary.

Keywords: Harish-Chandra bimodule, Rational Cherednik algebra, Category \( \mathcal{O} \).

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First of all, if \( p \cong \mathbb{C}^{[S/W]} \) denotes the space of all parameters for the Cherednik algebra, there is a lattice \( p_Z \subseteq p \) that generalizes the notion of integrality mentioned in the previous paragraph, see Subsection 2.6 for a precise definition. For a parameter \( c \), we construct in Subsection 5.2 a normal subgroup \( W_c \subseteq W \) satisfying the following properties:

1. \( W_c \) is a reflection group.
2. \( W_c = \{1\} \) if and only if \( c \in p_Z \). If \( c \) is generic, then \( W_c = W \).
3. \( W_c = W_c' \) provided \( c - c' \in p_Z \).

**Theorem 1.1.** Let \( W \) be a complex reflection group, and let \( c, c' \in \mathbb{C}[S] \) be conjugation invariant functions. The following is true.

1. The category \( \mathcal{HC}(c, c') \) is nonzero if and only if there exists a character \( \varepsilon : W \to \mathbb{C}^\times \) such that \( c - \varepsilon c' \in p_Z \).
2. If \( \mathcal{HC}(c, c') \) is nonzero, then it is equivalent to the category of representations of the group \( W/W_c \). If \( c = c' \), this is an equivalence of monoidal categories.

We study more closely the case where \( W = \mathfrak{S}_n \), the symmetric group in \( n \) elements. In this case, the possible supports of modules in category \( \mathcal{O} \) have been classified, [BE], and it is known which irreducibles have a given support, [Wi]. Using the results of loc. cit. together with Theorem 3.8 we are able to describe all irreducible HC bimodules over \( H_{c}(\mathfrak{S}_n) \), see Section 6. Our main result for type A is the following.

**Theorem 1.2.** Consider the rational Cherednik algebra \( H_{c}(\mathfrak{S}_n) \), where \( c = r/m \) with gcd\((r; m) = 1 \) and \( 1 < m \leq n \). The possible supports for HC bimodules form a chain \( (\mathfrak{h} \oplus \mathfrak{h}^*)/\mathfrak{S}_n = \mathbb{Z}_0 \supseteq \mathbb{Z}_1 \supseteq \cdots \supseteq \mathbb{Z}_{[n/m]} \).

For each \( i = 0, \ldots, [n/m] \), the category of \( HC \) \( c \)-bimodules supported on \( \mathbb{Z}_i \), modulo those supported on \( \mathbb{Z}_{i+1} \) is equivalent, as a monoidal category, to the category of representations of \( \mathfrak{S}_i \).

Theorem 1.2 is proved in Subsection 6.4 see Theorem 6.8. We also describe the category of HC bimodules over the algebra \( H_{c}(\mathfrak{S}_n) \), \( c = r/n \), with gcd\((r; n) = 1 \). In this case, the algebra \( H_{c}(\mathfrak{S}_n) \) admits a finite dimensional representation, and the category of HC bimodules has two irreducible objects: a finite dimensional representation, and the category of HC bimodules has two irreducible objects: a finite dimensional representation, and the closure of that subvariety. This is used in Theorem 1.1 to find a description of all (not only fully supported) irreducible HC bimodules over rational Cherednik algebras associated to symmetric groups. We also give a complete description of the category \( HC(H_{r/n}(\mathfrak{S}_n)) \), where gcd\((r; n) = 1 \). The proof of this is based on the vanishing of several extension groups.

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2. Preliminaries

2.1. Rational Cherednik Algebras. Fix a complex reflection group \( W \), let \( \mathcal{S} \subseteq W \) be the set of reflections and let \( c : \mathcal{S} \to \mathbb{C} \) be a conjugation invariant function. For each reflection \( s \in \mathcal{S} \), let \( \alpha_s \in \mathfrak{h}^* \) be an eigenvector with eigenvalue \( \lambda_s \neq 1 \). Also, let \( \alpha_s^\vee \in \mathfrak{h} \) be an eigenvector of \( s \) with eigenvalue \( \lambda_s^{-1} \). We remark that \( \alpha_s^\vee \), \( \alpha_s \) are unique up to multiplication by a nonzero scalar, and we normalize so that \( (\alpha_s, \alpha_s^\vee) = 2 \). The rational Cherednik algebra \( H_c := H_c(W, h) \) is the quotient of the smash product algebra \( T(\mathfrak{h} \oplus \mathfrak{h}^*) \# W \), where \( T(\cdot) \) denotes the tensor algebra, by the relations:

\[
[x, x'] = 0, \\
[y, x] = \langle y, x \rangle - \sum_{s \in \mathcal{S}} c(s) \langle \alpha_s, y \rangle \langle x, \alpha_s^\vee \rangle s \quad x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}.
\]

The algebra \( H_c \) is filtered, with \( W, \mathfrak{h}^* \) in filtration degree 0 and \( \mathfrak{h} \) in filtration degree 1. Its associated graded is the smash product \( \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# W \). Thus, we have a triangular decomposition \( H_c = \mathbb{C}[\mathfrak{h}] \otimes CW \otimes \mathbb{C}[\mathfrak{h}^*] \) given by the multiplication map, see e.g. [EG, Theorem 1.3].

Let \( \delta := \prod_{s \in \mathcal{S}} \alpha_s \). This is a \( W \)-semiinvariant element of \( \mathbb{C}[\mathfrak{h}] \). Let \( \mathfrak{h}^{reg} \) be the principal open set in \( \mathfrak{h} \) determined by \( \delta \). Note that \( \mathfrak{h}^{reg} \) coincides with the locus where the \( W \)-action is free. The operator \( [\delta, \cdot] \) is locally nilpotent, so the localization \( H_c[\delta^{-1}] \) makes sense, and it is isomorphic to the algebra \( D(\mathfrak{h}^{reg}) \# W \).

We will need the spherical rational Cherednik algebra which is constructed as follows. Let \( e := \frac{1}{|\mathcal{S}|} \sum_{w \in W} w \) be the trivial idempotent. The associated spherical subalgebra of \( H_c \) is \( eH_ce \). We remark that \( eH_ce \) inherits a filtration from that of \( H_c \), and \( \text{gr}(eH_ce) = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# W \). Note that \( e^{|\mathfrak{h}^*|} \in eH_ce \) for some \( n > 0 \), so that the localization \( eH_ce[(e^{|\mathfrak{h}^*|})^{-1}] \) makes sense and, moreover, it is isomorphic to \( D(\mathfrak{h}^{reg}/W) \). If \( c \) is a parameter such that the algebras \( H_c \) and \( eH_ce \) are Morita equivalent (that is, \( H_c = eH_ceH_c \)) then we say that \( c \) is spherical.

2.2. Homogeneous rational Cherednik algebras. Sometimes, see e.g. Section 3 below, it will be more convenient to work with the homogeneous version of the rational Cherednik algebra. Namely, let \( \mathcal{S} = \bigsqcup_{i=1}^r \mathcal{S}_i \) be the decomposition of \( \mathcal{S} \) into conjugacy classes, and let \( \mathfrak{h}, c_1, \ldots, c_r \) be independent variables. For \( s \in \mathcal{S}_i \), define \( c(s) := c_i \). Let \( c \) be the vector space with basis \( \mathfrak{h}, c_1, \ldots, c_r \). Then, \( \mathcal{H} \) is the \( S(c) \)-algebra defined by generators and relations analogous to those of the previous subsection, with the commutation relation between \( \mathfrak{h} \) and \( \mathfrak{h}^* \) replaced by

\[
[y, x] = h \langle y, x \rangle - \sum_{s \in \mathcal{S}} c(s) \langle \alpha_s, y \rangle \langle x, \alpha_s^\vee \rangle s,
\]

note that the algebra \( \mathcal{H} \) is graded, with \( W, \mathfrak{h}^* \) in degree 0 and \( \mathfrak{h}, c \) in degree 1. We remark that \( \mathcal{H} \) is a flat \( S(c) \)-algebra, see e.g. [L, Proposition 1.1.1], and that \( \mathcal{H}/c\mathcal{H} = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \# W \).

Let \( R_h(H_c) \) denote the Rees algebra of \( H_c \) with respect to the filtration described in the previous subsection. We have a quotient map \( \mathcal{H} \to R_h(H_c) \), given by \( w \mapsto w, x \mapsto x, y \mapsto hy, h \mapsto h, c_i \mapsto hc_i, w \in W, x \in \mathfrak{h}^*, y \in \mathfrak{h} \) and \( c_i := c(s) \) for \( s \in \mathcal{S}_i \). So we can pass from \( H_c \)-modules to \( \mathcal{H} \)-modules using the Rees construction with respect to some filtration, see for example Subsection 2.8.

2.3. Category \( \mathcal{O}_c \). The triangular decomposition \( H_c = \mathbb{C}[\mathfrak{h}] \otimes CW \otimes \mathbb{C}[\mathfrak{h}^*] \) allows us to define a category \( \mathcal{O}_c \) of modules over \( H_c, \) [CGOR]. By definition, \( \mathcal{O}_c \) is the full subcategory of the category of finitely generated \( H_c \)-modules consisting of those modules for which \( \mathfrak{h} \) acts by locally nilpotent endomorphisms. For example, finite dimensional representations, when they exist, belong to \( \mathcal{O}_c \). Also, for an irreducible representation \( \tau \) of \( W \), consider \( \tau \) as an \( S(\mathfrak{h}) \# W \)-module by letting \( \mathfrak{h} \) act on \( \tau \) by 0. Then, the Verma module \( \Delta_c(\tau) := \text{Ind}_{S(\mathfrak{h}) \# W}^{H_c} \tau = H_c \otimes_{S(\mathfrak{h}) \# W} \tau \) belongs to \( \mathcal{O}_c \). The Verma module \( \Delta_c(\tau) \) admits a unique irreducible quotient, \( L_c(\tau) \). Moreover, the set \( \{ L_c(\tau) : \tau \text{ an irreducible representation of } W \} \) forms a complete and irredundant collection of irreducible objects in \( \mathcal{O}_c \).
A module $M \in \mathcal{O}_c$ is finitely generated over the subalgebra $\mathbb{C}[h] \subseteq H_c$, so it can be viewed as a coherent sheaf on $h$. Hence, we may define its support $\text{supp}(M)$ as the support of $M$ as a $\mathbb{C}[h]$-module. This is a $W$-invariant subvariety of $h$. In fact, it is a union of strata of the stratification of $h$ with respect to stabilizers in $W$ of points in $h$. We remark that the support of an irreducible module $N \in \mathcal{O}_c$ is irreducible when viewed as a subvariety of $h/W$, see [BE] Proposition 3.22, and that a module $M \in \mathcal{O}_c$ is finite dimensional if and only if its support consists of only the 0 point.

### 2.4. Hecke algebras.

Let $B_W := \pi_1(\mathfrak{h}^{reg}/W)$ be the generalized braid group associated to $W$. The group $B_W$ admits a system of generators indexed by the set $A$ of reflection hyperplanes on $h$. For each $\Gamma \in A$, the pointwise stabilizer $W_\Gamma$ is cyclic, of order say $\ell_\Gamma$. Let $s_\Gamma \in S \cap W_\Gamma$ be the element with determinant $\exp(2\pi \sqrt{-1}/\ell_\Gamma)$, and let $T_\Gamma$ be a generator of the monodromy around $\Gamma$ such that a lift of $T_\Gamma$ to $\mathfrak{h}^{reg}$ is represented by a path from $x_0$ to $s_\Gamma(x_0)$, see [BMR] Appendix 1 for a precise definition. The set $\{T_\Gamma\}_{\Gamma \in A}$ is a generating set for the group $B_W$.

To define the Hecke algebra, for each reflection hyperplane $\Gamma \in A$, fix nonzero complex numbers $q_{\Gamma,0}, \ldots, q_{\Gamma,\ell_\Gamma - 1}$, in such a way that if $\Gamma, \Gamma'$ are $W$-conjugate then $q_{\Gamma,i} = q_{\Gamma',i}$ for each $i = 0, \ldots, \ell_\Gamma - 1$. We denote this collection of complex numbers by $h_\Gamma$. The Hecke algebra $H_q$ is, by definition, the quotient of the group algebra $CB_W$ by the relations $\prod_{\Gamma \in A} (T_\Gamma - q_{\Gamma,i})$, one for each $\Gamma \in A$. For example, setting $q_{\Gamma,i} = \exp(2\pi \sqrt{-1}/\ell_\Gamma)$ we recover the group algebra $CW$. We note that for each map $a : A \longrightarrow \mathbb{C}^\times$ that is constant on $W$-conjugacy classes we can rescale the parameters $q_{\Gamma,i} \mapsto a(\Gamma) q_{\Gamma,i}$ without changing the algebra, cf. [R] Subsection 3.3.3. Most of the time below, we will normalize so that $q_{\Gamma,0} = 1$ for all $\Gamma \in A$.

**Remark 2.1.** For the Hecke algebra of a Coxeter group, we use the normalization of the Hecke algebra $H_q$ whose quadratic relations read $(T_i - 1)(T_i + q) = 0$, which is not standard. Hence, our $T_i$’s differ from those of [DJ] by a factor of $q$, and our $q$ corresponds to $q^{-1}$ in [DJ]. Using this normalization, we say that an $H_q$-module is trivial if all $T_i$’s act by 1. See also Remark after Theorem 2.6 in [DJ].

### 2.5. KZ functor.

Consider the Cherednik algebra $H_c := H_c(W)$. In [GGOR] it is shown that there exists a quotient functor $KZ : \mathcal{O}_c \rightarrow \mathcal{H}_q(W)$-mod, the category of finite dimensional $\mathcal{H}_q(W)$-modules, where the parameter $q$ explicitly depends on $c$ and $\mathcal{O}_c$ denotes the unique non-trivial eigenvalue for the action of $s$ on $\mathfrak{h}^*$. For each reflection hyperplane $\Gamma \in A$, define

$$k_{\Gamma,i} := \sum_{s \in S \cap W_\Gamma} \frac{2c(s)(\lambda_{s^{-1}} - 1)}{1 - \lambda_s}, \quad i = 0, \ldots, \ell_\Gamma - 1.$$  \hspace{1cm} (2)

Note that $k_{\Gamma,i}$ depends only on the conjugacy class of $\Gamma$, and that $k_{\Gamma,0} = 0$. Now the parameter $q$ is computed as follows:

$$q_{\Gamma,i} := \exp(2\pi \sqrt{-1}(k_{\Gamma,i} - i)/\ell_\Gamma).$$  \hspace{1cm} (3)

Note that $q_{\Gamma,0} = 1$. Now we give the construction of the KZ functor. Start with a module $M \in \mathcal{O}_c$. Then, the localization $M[\delta^{-1}]$ is a $D(\mathfrak{h}^{reg})\#W$-module. Since $W$ acts freely on $\mathfrak{h}^{reg}$, there is an equivalence of categories $D(\mathfrak{h}^{reg})\#W$-mod $\rightarrow D(\mathfrak{h}^{reg}/W)$-mod, given by $M \mapsto eM$. Since $M \in \mathcal{O}_c$, is a $\mathbb{C}[h]$-coherent module, we have that $eM[\delta^{-1}]$ must be a local system. Hence, $DR(eM[\delta^{-1}])$ is a representation of the braid group $B_W$, where $DR$ stands for the de Rham functor, see e.g. [HTT], Chapter 7. By [GGOR Theorem 5.13], the action of $CB_W$ on $DR(eM[\delta^{-1}])$ factors through the Hecke algebra $H_q$. We abbreviate $KZ(M) := DR(eM[\delta^{-1}])$.

For example, let $\tau$ be a 1-dimensional representation of $W$. For a hyperplane $\Gamma \in A$, let $s_\Gamma$ be a generator of $W_\Gamma$ with $\lambda_{s^{-1}} = \exp(2\pi \sqrt{-1}/\ell_\Gamma)$. Denote by $C_{\Gamma,i}$ the 1-dimensional representation of $W_\Gamma$ where $s_\Gamma$ acts by $\lambda_{s^{-1}}$. In particular, we have that $\text{Res}_{W_\Gamma}^W \tau = C_{\Gamma,\tau(\Gamma)}$ for $\tau(\Gamma) \in \{0, \ldots, \ell_\Gamma - 1\}$. Then, we have that $KZ(\Delta(\tau))$ is the 1-dimensional $\mathcal{H}_q$-module where $T_\Gamma$ acts by $q_{\Gamma,\tau(\Gamma)}$. In particular, $KZ(\text{triv})$ is the trivial representation of $\mathcal{H}_q$.

We remark that the local system $eM[\delta^{-1}]$ has regular singularities, see e.g. [GGOR Section 5.3]. It follows, in particular, that the KZ functor is exact. Moreover, the essential image of KZ consists of all finite dimensional modules over the algebra $\mathcal{H}_q$, [L2]. The kernel of $KZ, \mathcal{O}_c^{tor}$, consists of those modules $M$
for which supp(M) is properly contained in h. This is a Serre subcategory of Oc. Then, KZ induces an equivalence Oc/oc ≅ Hq-mod where, recall, the latter category is that of finite dimensional Hq-modules.

2.6. Lattice pZ. Let p∗ := c/h ≅ C[S/W], a vector space with basis c1,...,cr. We remark that the set of parameters ‘c’ for the Cherednik algebra Hc can be naturally identified with p, the dual of p∗. So we can view kT,i as an element of p∗, its value on a parameter c ∈ p is given by the formula [2]. Define pZ to be the Z-lattice inside p∗ spanned by elements ℓ−1(kT,i − kT,j), and let pZ ⊆ p be the dual lattice. The lattice pZ consists of all parameters c such that Hq is naturally isomorphic to CW, and thus the set of parameters for the Hecke algebra can be identified with p/pZ, see [LA Subsection 2.6]. For example, if W is a Coxeter group, then pZ coincides with the set of parameters for which c(s) ∈ Z for all s ∈ S.

We will need a spanning set for pZ. First, let us introduce some notation. For each Γ ∈ A, the set of characters of WΓ is identified with Z/ℓΓZ, an isomorphism Z/ℓΓZ −→ Hom(WΓ,C∗) is given by m −→ (s −→ det(s)m). We have a morphism Hom(W,C∗) −→ ΠΓ∈A/W Hom(WΓ,C∗), given by restriction. This is an isomorphism, [R Subsection 3.3.1]. Thus, we have a correspondence between 1-dimensional characters χ of W and |A/W|-tuples of integers (mΓ) with 0 ≤ mΓ ≤ ℓΓ − 1. So, to a character χ ∈ Hom(W,C∗) associated to the tuple (mΓ) we assign χ ∈ p, given by

\[ \overline{\chi}(kT,i) = \begin{cases} \ellT - mT & \text{if } i \geq \ell - mT \\ -mT & \text{if } \ell < \ell - mT. \end{cases} \]

Clearly, χ ∈ pZ, and the elements χ form a spanning set for pZ.

Let us explain the reason why we are interested in the elements χ. Recall that we have an embedding Hc ↪ D(h∗)W. We have an automorphism of D(h∗)W, given by w −→ χ(w)w, x −→ x, ∂y −→ ∂y, w ∈ W, x ∈ h∗, y ∈ h. Then, according to [BC Section 5.1], under this automorphism Hc transforms to Hc+Γ, while eHc,e transforms to eχHc+χ∂. Here, eχ denotes the idempotent corresponding to χ, eχ = |W|−1 \[ \sum \chi(w⁻¹)w \]. We will use this in Subsection 5.3 see in particular Lemma 5.5.

2.7. Restriction functors for category Oc. We remark that, if W′ is a parabolic subgroup of W (that is, the stabilizer of a point in h) then there is a natural inclusion of algebras : Hq(W′) ⊂ Hq(W) where, abusing the notation, we also denote by q its restriction to S ∩ W′. This allows us to define a restriction functor HResW′ := e∗ : Hq(W′)-mod → Hq(W)-mod.

There is also a restriction functor on the level of category Oc, [BE]. This functor depends on the choice of a point b ∈ h whose stabilizer Wb coincides with W′. For distinct b,b′ with this property, the functors are isomorphic (but not canonically so) so we will just denote this functor by ResW′. This functor is defined as follows. Let b ∈ h be a point with Wb = W′. We will denote also by b its projection to h/W. Consider C[h/W]^h, the completion of C[h/W] at the maximal ideal of h. Then it can be easily checked that Hc^h := C[h/W]^h ⊗ C[h/W] Hc is naturally an algebra. Bezrukavnikov and Etingof showed in [BE Theorem 3.2] that Hc^h is isomorphic to a matrix algebra of size |W/W′| with coefficients in Hc(W′,h)^h, see Subsection 3.2 for a more precise statement. For a module M in category Oc, let E(M^h) denote the h-locally nilpotent part of M^h where, abusing the notation, we denote by M^h the image of the completion C[h/W]^h ⊗ C[h/W] M under an equivalence Hc^h-mod → Hc(W′,h)^h-mod. Then, ResW′(M) is defined to be \{ χ ∈ E(M^h) : yv = 0 for all y ∈ h/W′ \}. This is a module in category Oc for Hc(W′,h/hW′), cf. [BE Section 2].

We will also need a certain compatibility between the restriction functors and the KZ functor that was established in [Sh Section 2]. Namely, let W′ be a parabolic subgroup of W. Then, as is checked in [Sh Theorem 2.1], we have KZ ◦ ResW′ = HResW′ ◦ KZ, where KZ denotes the KZ functor from category Oc(W′,h/hW′) to Hq(W′).

2.8. Harish-Chandra bimodules. By a HC Hc-Hc-bimodule we mean a finitely generated Hc-Hc-bimodule such that the adjoint action of every element of C[h]^W ⊔ C[h∗]^W is locally nilpotent. For example, the algebra Hc is always a HC Hc-bimodule. Clearly, the category HC(c,c′) of HC Hc-Hc-bimodules is a Serre subcategory of the category of all Hc-Hc-bimodules. The following proposition gives basic properties of HC bimodules. For its proof, see e.g. Lemma 3.3 in [BEQ], or Proposition 3.1 in [L4].
Proposition 2.2. \(1\) Any \(B \in \text{HC}(c, c')\) is finitely generated as a left \(H_c\)-module, as a right \(H_{c'}\)-module, and as a \(\mathbb{C}[h]^W \otimes \mathbb{C}[h^*]^W\)-module (here, \(\mathbb{C}[h]^W\) is considered inside \(H_c\), while \(\mathbb{C}[h^*]^W\) is considered inside \(H_{c'}\)).

\(2\) If \(B \in \text{HC}(c, c')\), \(B' \in \text{HC}(c', c'')\) then \(B \otimes_{H_{c'}} B' \in \text{HC}(c, c'')\).

\(3\) If \(B \in \text{HC}(c, c')\) and \(M \in \mathcal{O}_c\), then \(B \otimes_{H_{c'}} M \in \mathcal{O}_c\).

A way to construct HC bimodules is as follows. Consider modules \(N \in \mathcal{O}_c\), \(M \in \mathcal{O}_{c'}\). Then, \(\text{Hom}_{H_c}(M, N)\) is an \(H_c\)-\(H_{c'}\)-bimodule. By \(\text{Ann}(M, N)\) we mean the sub-bimodule of \(\text{Hom}_{H_c}(M, N)\) consisting of all those vectors that are locally nilpotent under the adjoint action of \(\mathbb{C}[h]^W \cup \mathbb{C}[h^*]^W\). Clearly, \(\text{Ann}(M, N)\) is the direct limit (= union) of its HC sub-bimodules. It was checked in [L] Proposition 5.7.1 that \(\text{Ann}(M, N)\) is actually HC. We remark, [L] Lemma 5.7.2, that if \(M\) and \(N\) are irreducible then \(\text{Ann}(M, N) = 0\) unless \(\text{supp}(M) = \text{supp}(N)\). Also, note that \(\text{Ann}(M, N) \neq 0\) if and only if there exists a HC \(H_c\)-\(H_{c'}\)-bimodule and a nonzero morphism of left \(H_c\)-modules \(B \otimes M \rightarrow N\).

Proposition 2.3. Let \(B\) be an irreducible HC \(H_c\)-\(H_{c'}\)-bimodule. Then, there exist irreducible modules \(M \in \mathcal{O}_{c'}\), \(N \in \mathcal{O}_c\) and a monomorphism \(B \hookrightarrow \text{Ann}(M, N)\).

Proof. By [L] Lemma 3.10, there exists a nonzero module \(M \in \mathcal{O}_{c'}\) with \(B \otimes_{H_{c'}} M \neq 0\). Since the latter module is in category \(\mathcal{O}_c\), there exists an irreducible module \(N \in \mathcal{O}_c\) and a nonzero map \(f : B \otimes_{H_{c'}} M \rightarrow N\). Then, \(v \mapsto (m \mapsto f(v \otimes_{H_{c'}} m))\) defines a nonzero morphism \(B \rightarrow \text{Ann}(M, N)\).

An equivalent definition of HC bimodules was found in [L] Section 3. Namely, recall that the algebras \(H_c, H_{c'}\) are filtered and there is a natural identification \(\text{gr} \hspace{0.01in} \mathcal{H}_c = \mathcal{H}_{c'} = \mathbb{C}[h \otimes h^*]/\#W\), see Subsection 2.1. Let \(B\) be a filtered \(H_c\)-\(H_{c'}\)-bimodule. Then, \(\text{gr} \hspace{0.01in} B\) is an \(\mathbb{C}[h \otimes h^*]/\#W\)-bimodule. It is proved in [L] Subsection 5.4, that \(B\) is HC if and only if it admits a bimodule filtration such that \(\text{gr} \hspace{0.01in} B\) is a finitely generated \(\mathbb{C}[h \otimes h^*]/\#W\)-bimodule and, moreover, the left and right actions of \(\mathbb{C}[h \otimes h^*]/W\) on \(\text{gr} \hspace{0.01in} B\) coincide. We call such a filtration on \(B\) good. Note that this implies that \(\text{gr} \hspace{0.01in} B\) is a finitely generated module over \(\mathbb{C}[h \otimes h^*]/W\). Then, we define the (singular) support of \(B\), \(SS(B) \subseteq (h \otimes h^*)/W\) to be the support of \(\text{gr} \hspace{0.01in} B\) as a \(\mathbb{C}[h \otimes h^*]/W\)-module. We remark that, as usual, \(\text{gr} \hspace{0.01in} B\) depends on the filtration, but its support does not. We also remark that \(SS(B)\) is a Poisson subvariety of \((h \otimes h^*)/W\) and, moreover, it is a union of symplectic leaves. The variety \((h \otimes h^*)/W\) has finitely many symplectic leaves, see e.g. [BrGo] Subsection 7.4.

Lemma 2.4 ([L], Lemma 4.2). Let \(B\) be a HC \(H_c\)-\(H_{c'}\)-bimodule. Then, \(SS(B) = SS(H_c/\text{LAnn}(B)) = SS(H_{c'}/\text{LAnn}(B))\), where \(\text{LAnn}(B)\), \(\text{RAann}(B)\) denote the left and right annihilator of \(B\), respectively, and \(H_c/\text{LAnn}(V)\) (resp. \(H_{c'}/\text{LAnn}(V)\)) is viewed as a HC \(H_c\)-bimodule (resp. HC \(H_{c'}\)-bimodule).

Lemma 2.5. Let \(B\) be a finitely generated irreducible HC \(H_c\)-\(H_{c'}\)-bimodule, and let \(N \in \mathcal{O}_{c'}\) be irreducible. Then, \(B \otimes_{H_{c'}} N \neq 0\) unless \(\text{RAann}(B) = \text{Ann}(N)\). If \(B \otimes_{H_{c'}} N \neq 0\), then the annihilator of every irreducible quotient of \(B \otimes_{H_{c'}} N\) coincides with \(\text{LAnn}(B)\). Moreover, \(\text{LAnn}(B) = \text{Ann}(B \otimes_{H_{c'}} N)\).

Proof. Assume \(B \otimes_{H_{c'}} N \neq 0\), and let \(M\) be an irreducible quotient of \(B \otimes_{H_{c'}} N\). Then, as in Proposition 2.3, we have an inclusion \(B \hookrightarrow \text{Hom}_{H_{c'}}(N, M)\). First, we check that \(\text{RAann}(B) = \text{Ann}(N)\). Note that \(\bigcap_{f \in B} \ker(f)\) is a proper submodule of \(N\). Since \(N\) is irreducible, we must have \(\bigcap_{f \in B} \ker(f) = 0\). Now, if \(a \in \text{RAann}(B)\), then \(aN \subseteq \bigcap_{f \in B} \ker(f)\), so \(a \in \text{Ann}(N)\). The other inclusion is clear. To show that \(\text{LAnn}(B) = \text{Ann}(M)\), we observe that \(\bigcap_{f \in B} f(N)\) is a nonzero submodule of \(M\), so we must have \(\bigcap_{f \in B} f(N) = M\). From here it follows easily that \(\text{LAnn}(B) = \text{Ann}(M)\). To prove the last statement of the lemma, note that we clearly have \(\text{LAnn}(B) \subseteq \text{Ann}(B \otimes_{H_{c'}} N)\). On the other hand, by what we just proved \(\text{Ann}(B \otimes_{H_{c'}} N) \subseteq \text{Ann}(M) = \text{LAnn}(B)\). So \(\text{LAnn}(B) = \text{Ann}(B \otimes_{H_{c'}} N)\).

Corollary 2.6. Let \(B_1\) be an irreducible HC \(H_c\)-\(H_{c'}\)-bimodule, and \(B_2\) an irreducible HC \(H_{c'}\)-\(H_{c''}\)-bimodule. Then, \(B_1 \otimes_{H_{c'}} B_2 = 0\) unless \(SS(B_1) = SS(B_2)\).

Proof. Assume that \(SS(B_1) \neq SS(B_2)\), and denote \(B := B_1 \otimes_{H_{c'}} B_2\). First, we assume that \(SS(B_1) \subseteq SS(B_2)\). By [L] Lemma 3.10 it is enough to show that \(B \otimes_{H_{c''}} N = 0\) for all irreducible modules \(N \in \mathcal{O}_{c''}\). If \(B_2 \otimes_{H_{c''}} N = 0\) we are done. So we may assume that \(B_2 \otimes_{H_{c''}} N \neq 0\). By the previous lemma this implies that \(\text{Ann}(B_2 \otimes_{H_{c''}} N) = \text{LAnn}(B_2)\). If \(B_1 \otimes_{H_{c'}} (B_2 \otimes_{H_{c''}} N) \neq 0\), then \(B_1 \otimes_{H_{c'}} M \neq 0\) for some irreducible subquotient \(M\) of \(B_2 \otimes_{H_{c''}} N\). So \(\text{RAann}(B_1) = \text{Ann}(M) \supseteq \text{Ann}(B_2 \otimes_{H_{c''}} N) = \text{LAnn}(B_2)\). Thus,
SS(B₁) = SS(H_c / RAnn(B₁)) ⊆ SS(H_c / LAnn(B₂)) = SS(B₂), a contradiction with our assumption. We conclude that B₁ ⊗_{H_c} B₂ = 0.

Now assume that SS(B₂) ⊄ SS(B₁). Let e^{opp} be defined by e^{opp}(s) := -c(s^{-1}). Then, it is easy to check that we have an isomorphism H_e(W, h) → H_e^{opp}(W, h^*)^{opp} given by x → x, y → y, w → w^{-1}, x ∈ h^*, y ∈ h, w ∈ W. We get an equivalence ρ_{c,e} : H_c-H_c-bimodule → H_e^{opp}(W, h^*)-H_e^{opp}(W, h^*)-bimodule. Similarly, we get equivalences ρ_{c,e',c}, ρ_{c,e''}. Note that these equivalences preserve the categories of HC bimodules as well as the support of a HC bimodule. We have that ρ_{c,e}′(B₁ ⊗_{H_c} B₂) = ρ_{c,e''}(B₂) ⊗_{H_e^{opp}(W, h^*)} ρ_{c,e'}(B₁). Thus, the result in this case follows from the previous paragraph.

The definition of a HC bimodule given in [L, Section 3] also allows us to give a definition of a HC bimodule in such a way that, if B is a HC H_c-H_c-bimodule with a good filtration, then R_h(B) is a HC H-bimodule (recall that R_h(H_c), R_h(H_c) are quotients of H).

Definition 2.7. A HC H-bimodule B is a finitely generated, graded H-bimodule satisfying the following conditions:

(i) The left and right actions of h on B coincide.

(ii) B is flat as a C[h]-module.

(iii) The left and right actions of Z(H/hH) on B/hB coincide.

We remark that we also have a notion of Harish-Chandra bimodule for spherical rational Cherednik algebras. In fact, the definition does not change, because the parameter bimodule has a unique dense symplectic leaf.

4.1] proves the following.

• HC functor factors through the quotient HC H_c-bimodule with a good filtration, then RH(B) is a HC H-bimodule (recall that RH(H_c), RH(H_c) are quotients of H).

Definition 2.7. A HC H-bimodule B is a finitely generated, graded H-bimodule satisfying the following conditions:

(i) The left and right actions of h on B coincide.

(ii) B is flat as a C[h]-module.

(iii) The left and right actions of Z(H/hH) on B/hB coincide.

We remark that we also have a notion of Harish-Chandra bimodule for spherical rational Cherednik algebras. In fact, the definition does not change, because the parameter bimodule has a unique dense symplectic leaf.

Using restriction functors and the fact that (h^*h^*)/W has a unique dense symplectic leaf L, which coincides with the projection of the set of points in h^* with
trivial stabilizer. The category $\text{HC}_{\mathcal{L}}(H_c)$ is the quotient of the category of all HC $H_c$-bimodules modulo the Serre subcategory formed by HC bimodules with proper support, and we denote $\overline{\text{HC}}(H_c) := \text{HC}_{\mathcal{L}}(H_c)$.

**Proposition 2.9.** The category $\overline{\text{HC}}(c, c)$ is semisimple.

**Proof.** Pick a point $x \in \mathfrak{h} \oplus \mathfrak{h}^*$ whose stabilizer in $W$ is trivial, its projection to $(\mathfrak{h} \oplus \mathfrak{h})^*/W$ is a point in the open symplectic leaf $\mathcal{L}$. Note that $\Xi = W$, and $H_c = \mathbb{C}$, so $\text{HC}_0^\mathcal{L}(H_c)$ is precisely the category of finite dimensional representations of $W$. The results from [L] mentioned above imply that $\overline{\text{HC}}(c, c)$ can be embedded as a full subcategory of the category of representations of $W$. Moreover, this subcategory is closed under subquotients and tensor products. It follows that $\overline{\text{HC}}(c, c)$ is equivalent to the category of representations of $W/N$ for a normal subgroup $N \subseteq W$. In particular, it is a semisimple category. □

In Section 5, we will find an explicit description of the subgroup $N$ that appears in the proof of Proposition 2.9, see Subsection 5.2. We will also see, Corollary 5.2, that $\overline{\text{HC}}(H_c, H_c)$ is semisimple for different parameters $c, c'$. Another application of restriction functors is the following result.

**Proposition 2.10.** The regular bimodule $H_c$ is injective in the category of HC $H_c$-bimodules.

**Proof.** In view of Proposition 2.8, we need to show that $\text{Ext}(B, H_c) = 0$ for any irreducible HC bimodule $B$, where $\text{Ext}$ denotes $\text{Ext}_{H_c}^1$. We separate in two cases.

**Case 1:** $B$ has proper support. This case is contained in an old version of the paper [BL]. We provide a proof for the reader’s convenience. Consider an exact sequence $0 \to H_c \to X \to B \to 0$. Let $x$ be in the open symplectic leaf of $(\mathfrak{h} \oplus \mathfrak{h}^*)/W$, and consider the corresponding restriction functor $\bullet_x := \bullet_{x,0}$. Note that $B_x = 0$. Since the restriction functor is exact, we must then have $((H_c)_x)^\dagger = (X_x)^\dagger$. We have the adjunction map $X \to ((H_c)_x)^\dagger$. The latter bimodule admits a filtration whose associated graded is contained in $((\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*/#W])^\dagger_0$, see [L] Subsection 3.6 for a construction of the functor $\bullet_x$ for the algebra $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*/#W]$. By construction, $((\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*/#W])^\dagger_0$ is the global sections of the restriction of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*/#W]$ to the open symplectic leaf of $(\mathfrak{h} \oplus \mathfrak{h}^*)/W$. But the complement of this leaf has codimension 2. Hence, $((\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*/#W])^\dagger_0 = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*/#W]$, and this implies that $((H_c)_x)^\dagger = H_c$. Now the adjunction map $X \to H_c$ is a splitting of the exact sequence $0 \to H_c \to X \to B \to 0$.

**Case 2:** $B$ has full support. Assume $0 \to H_c \xrightarrow{\rho} X \to B \to 0$ is an exact sequence. Pick $x$ in the open symplectic leaf of $(\mathfrak{h} \oplus \mathfrak{h}^*)/W$, and consider the corresponding restriction functor $\bullet_x$. We have an exact sequence $0 \to (H_c)_x \to (X)_x \to B_x \to 0$. Since the category of $\Xi$-equivariant Harish-Chandra $H_c$-bimodules is semisimple, Proposition 2.9, this exact sequence splits, $(X)_x = (H_c)_x \oplus B_x$. Now, recall from the previous case that $(H_c)_x)^\dagger = H_c$, and that we have the adjunction morphism $X \to (X)_x^\dagger = H_c \oplus (B_x)^\dagger$. By [L] Theorem 3.7.3, the kernel of this morphism is a HC bimodule with proper support, so the morphism must be injective. Thus, we can consider $X \subseteq H_c \oplus (B_x)^\dagger$, and $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_1 : H_c \to H_c, \varphi_2 : H_c \to (B_x)^\dagger$. Let us remark that, since the center of $H_c$ is trivial, every nonzero endomorphism of $H_c$ is an automorphism. So, if $\varphi_2 \neq 0$, we can find a splitting for $\varphi$. Thus, we may assume $\varphi_1 = 0$, and $\varphi_2 : H_c \to (B_x)^\dagger$ is an inclusion.

Now recall that we have the adjunction morphism $B \to (B_x)^\dagger$. Since $B$ is irreducible, this is actually an injection. The cokernel of this morphism is a HC bimodule with proper support, this follows from [L] Proposition 3.7.3. Thus, we must have $B \subseteq \varphi_2(H_c)$, so $B$ is isomorphic to an ideal of $H_c$. But this implies that $B_x = (H_c)_x$. So the exact sequence $0 \to H_c \to X \to B \to 0$ induces an inclusion $X \subseteq (X)_x^\dagger = H_c \oplus H_c$ and, reasoning as in the previous paragraph, we can find a splitting for $\varphi$. Thus, $\text{Ext}(H_c, B) = 0$. □

We now explain a way to construct $\bullet_x$ that is convenient for us. We follow [L4] Section 3.2. First, we explain how to construct $\bullet_x$ for the homogeneous rational Cherednik algebra. Let $\mathcal{L} \subseteq \mathfrak{h}^{reg-W}/W'$ be the projection of $\{ x \in \mathfrak{h} : W_x = W' \} \subseteq \mathfrak{h}$. Here, $\mathfrak{h}^{reg-W'}$ denotes the principal open set $\{ x \in \mathfrak{h} : W_x \subseteq W' \} = \mathfrak{h} \setminus \bigcup_{s \in W} \Gamma_s$. Note that $\mathcal{L}$ is closed in $\mathfrak{h}^{reg-W}/W'$. We consider the completion $H^{\mathcal{L}}_{reg-W} := \mathbb{C}[\mathfrak{h}^{reg-W}/W']^{\mathcal{L}} \otimes_{\mathbb{C}[\mathfrak{h}/W]} \mathbb{H}$, which is naturally an algebra. For a HC $\mathbb{H}$-bimodule $B$, let $B^{\mathcal{L}}_{reg-W} := \mathbb{C}[\mathfrak{h}^{reg-W}/W']^{\mathcal{L}} \otimes_{\mathbb{C}[\mathfrak{h}/W]} B$. This is a $\Xi$-equivariant $H^{\mathcal{L}}_{reg-W}$-bimodule where, recall, we denote $\Xi = N_W(W')/W'$. The latter algebra is isomorphic to the algebra of $[W/W'] \times [W/W']$-matrices over $H(W', \mathfrak{h})^{\mathcal{L}}_{reg-W'}$, so we have a Morita equivalence between $H^{\mathcal{L}}_{reg-W}$ and $H(W', \mathfrak{h})^{\mathcal{L}}$. Abusing the notation, let us denote also by $B^{\mathcal{L}}_{reg-W}$, the corresponding $H(W', \mathfrak{h})^{\mathcal{L}}_{reg-W'}$-bimodule. Now let $B_\varnothing$ be the
subspace of $\mathbf{B}_{reg-W'}$ consisting of elements that commute with $\mathfrak{h}^{W'},(\mathfrak{h}^*)^{W'}$ and for which the action of $S(\mathfrak{h}^{W'})^{W'},S(\mathfrak{h}^{W'})^{W'}$ is locally nilpotent. Then, according to \cite[Lemma 3.7]{L4}, we have an isomorphism of functors $\bullet_{\reg} \cong \circ$. A more precise statement is as follows. Consider the functor $\mathcal{G} : \mathcal{HC}(\mathfrak{h}(W',\mathfrak{h})) \to \mathcal{HC}(\mathfrak{h}(W',\mathfrak{h}),\mathfrak{h}_{reg-W'})$ given by $\mathcal{G}(B') = \mathbb{C} [\mathcal{L} \times \mathfrak{h}^W/\mathfrak{h}^W] \mathcal{L} \otimes_{\mathbb{C}[\mathfrak{h}^W/\mathfrak{h}^W]} (D_h(\mathcal{L}) \otimes B')$. Then, $\mathcal{G}$ is a fully faithful embedding and $\bullet_{\reg}$ coincides with $\mathcal{G}^{-1}(\mathcal{L}^{\mathcal{H}}_{reg-W'})$ (the statement here includes that this latter functor is well-defined).

To construct $\bullet_{\reg}$ for rational Cherednik algebras $H_c,H_c'$ we use the Rees construction. Namely, let $B$ be a HC $H_c-H_c'$-bimodule, and pick a bimodule filtration on $B$ in a way that $R_0(B)$ is a HC $H$-bimodule. Then, $B_\ell := (R_\ell(B_{\reg})) / (\mathfrak{h} - 1)$ is a HC $H_{\ell}^{-}-H_{\ell}^{-}$-bimodule. The construction of $B_1$ does not depend, up to a distinguished isomorphism, on the choice of a filtration on $B$, see e.g. \cite[Subsection 3.9]{L4}.

Finally, let us state a compatibility result between restriction functors for category $\mathcal{O}$ and restriction functors for HC bimodules. Recall that for a HC $H_c-H_c'$-bimodule $B$ and $N \in \mathcal{O}_{c'}$, we have that $B \otimes_{H_c} N \in \mathcal{O}_c$. Then, \cite[Subsection 3.3]{L4}, we have

**Lemma 2.11.** There is a natural isomorphism $\text{Res}_{W'}^W(B \otimes_{H_c} N) \xrightarrow{\cong} B_{\reg} \otimes_{H_c} \text{Res}_{W'}^W(N)$.

### 3. Reduction to corank 1

**3.1. Localization.** Let $f \in \mathbb{C}[h]^W$. Since the adjoint action of $f$ on $\mathfrak{h}$ is locally nilpotent, the localization $\mathfrak{h}^{f^{-1}}$ makes sense as an algebra. We remark that $\mathfrak{h}^{f^{-1}}/\mathfrak{h}^W = \mathbb{C}[\pi^{-1}(U) \times \mathfrak{h}^*] \# W$. Here, $U$ denotes the principal open set in $\mathfrak{h}/W$ determined by $f$, and $\pi : \mathfrak{h} \to \mathfrak{h}/W$ is the natural projection. Note that the algebra $\mathfrak{h}^{f^{-1}}$ is graded, this follows because $f \in \mathfrak{h}$ is in degree 0. Also, note that $\mathfrak{h}^{f^{-1}} = \mathbb{C}[U] \otimes_{\mathbb{C}[h]/W} \mathfrak{h}$.

**Lemma 3.1.** Let $B$ be a HC $\mathfrak{h}$-bimodule, and $f \in \mathbb{C}[h]^W$. Then, all localizations $\mathbb{C}[U] \otimes_{\mathbb{C}[h]/W} B, B \otimes_{\mathbb{C}[h]/W} \mathbb{C}[U], \mathbb{C}[U] \otimes_{\mathbb{C}[h]/W} B \otimes_{\mathbb{C}[h]/W} \mathbb{C}[U]$ coincide.

**Proof.** Recall that $B$ is graded. Now, since the adjoint action of $f$ on $B/hB$ is 0, we have that $fv - vf \in hB$ for every $v \in B$. So we can define the operator $\frac{1}{f}[\cdot,\cdot]$ because $B$ is $\mathbb{C}[h]$-flat. Since $f$ has degree 0, this operator has degree $-1$. So the operator $\frac{1}{f}[\cdot,\cdot]$, and hence $[\cdot,\cdot]$, is locally nilpotent. The result follows. □

For a HC $\mathfrak{h}$-bimodule, $B$, we will denote by $B^{f^{-1}}$ any of the localizations of Lemma 3.1. Note that we can define the notion of a HC bimodule over $\mathfrak{h}^{f^{-1}}$ similarly to Definition 2.7. Clearly, $B^{f^{-1}}$ is a HC $\mathfrak{h}^{f^{-1}}$-bimodule.

We remark that, more generally, for a smooth affine algebraic variety $U$ with an étale map $U \to h/W$, the space $\mathbb{C}[U] \otimes_{\mathbb{C}[h]/W} \mathfrak{h}$ is actually an algebra. Indeed, it can be identified with the $S(c)$-subalgebra of $D_h(U \times h/W) \mathfrak{h}$ generated by $\mathbb{C}[U \times h/W] \mathfrak{h}$, $\mathfrak{c}$, and the Dunkl-Opdam operators. We will denote this algebra by $\mathfrak{h}_U$. Note that $\mathfrak{h}_U$ is graded, with $\mathbb{C}[U \times h/W] \mathfrak{h}$ in degree 0. We can define the notion of a HC $\mathfrak{h}_U$-bimodule as before. Note that we still have a notion of the support of a HC $\mathfrak{h}_U$-bimodule $B$: this is the support of $B/cB$ as a $Z(\mathfrak{h}_U/c\mathfrak{h}_U)$-module. If, moreover, $U \to h/W$ is an inclusion then, similarly to Lemma 3.1, for a HC $\mathfrak{h}_U$-bimodule $B$, all localizations $\mathbb{C}[U] \otimes_{\mathbb{C}[h]/W} B, B \otimes_{\mathbb{C}[h]/W} \mathbb{C}[U]$ and $\mathbb{C}[U] \otimes_{\mathbb{C}[h]/W} B \otimes_{\mathbb{C}[h]/W} \mathbb{C}[U]$ coincide. We will denote this by $B_U$. This is a HC $\mathfrak{h}_U$-bimodule.

**3.2. Bezrukavnikov-Etingof isomorphisms.** We will need some isomorphisms of étale lifts, that are essentially due to Bezrukavnikov and Etingof. \cite[Theorem 3.2]{BE}, see also \cite[Lemma 2.1]{L2}, \cite[Proposition 2.6]{W3}.

Let $W' \subseteq W$ be a parabolic subgroup. Recall that we have defined $\mathfrak{h}^{reg-W'} := \{ b \in \mathfrak{h} : W_b \subseteq W' \} = \mathfrak{h} \setminus \mathfrak{g}_{\emptyset} \cap \ker \alpha_s$. In particular, $\mathfrak{h}^{reg-W'}$ is a principal open set in $\mathfrak{h}$. We remark that the unramified locus of the map $\mathfrak{h}/W' \to \mathfrak{h}/W$ coincides with $\mathfrak{h}^{reg-W'}/W'$, so we have an étale morphism $\mathfrak{h}^{reg-W'}/W' \to h/W$ and the algebra $\mathfrak{h}_{reg-W'} := \mathfrak{h}^{reg-W'}/W'$ makes sense.

On the other hand, let $\mathfrak{h}$ be the rational Cherednik algebra for the action of $W'$ on $\mathfrak{h}$. We remark that we take this as an algebra over $S(c)$, even if the defining relations do not involve all the variables $c_1,\ldots,c_r$. We have a decomposition $\mathfrak{h} = \mathfrak{h}^{W'} \otimes \mathfrak{h}_{W'}$. Recall that here, $\mathfrak{h}^{W'}$ denotes the subspace of $W'$-invariants, and $\mathfrak{h}_{W'}$ a unique $W'$-stable complement to $\mathfrak{h}^{W'}$ in $\mathfrak{h}$. This induces a decomposition $\mathfrak{h} = \mathfrak{h}^+ \otimes_{\mathbb{C}[\mathfrak{h}]} D_h(\mathfrak{h}^W)$, where $\mathfrak{h}^+$ is the rational Cherednik algebra for the action of $W'$ on $\mathfrak{h}_{W'}$ (again, we include all variables $c_1,\ldots,c_r$).
We can form the centralizer algebra $Z(W, W', \mathbf{H}_{\text{reg}^\ast - W'})$. Recall that, by definition, for an algebra $A$ and a monomorphism $\mathbb{C}W' \hookrightarrow A$, we can form the right $A$-module $\text{Fun}_{W'}(A) := \{f : W \to A : f(w') = w'f(w)\}$, for all $w' \in W'$. This is a free right $A$-module of rank $|W/W'|$. Define $Z(W, W', A) := \text{End}_A(\text{Fun}_{W'}(W, A))$. We remark that $Z(W, W', A) \cong \text{Mat}_{W/W'}(A)$, but this isomorphism is not canonical. There is, however, a canonical way to recover $A$ from $Z(W, W', A)$. Namely, consider the element $e(W') \in Z(W, W', A)$ defined by

$$
[e(W')f](w) = \begin{cases} f(w) & \text{if } w \in W' \\ 0 & \text{else.} \end{cases}
$$

Then, $e(W')Z(W, W', A)e(W')$ is naturally identified with $A$.

**Lemma 3.2** ([BE], Theorem 3.2). There is an isomorphism

$$
\Theta : \mathbf{H}_{\text{reg}^\ast - W'} \to Z(W, W', \mathbf{H}_{\text{reg}^\ast - W'}). 
$$

We remark that [BE, Theorem 3.2] works with completions rather than étale lifts. However, $\mathbf{h}^\ast_{\text{reg}^\ast - W'}$ is defined as the complement of the reflection hyperplanes for reflections not in $W'$, so the proof of [BE, Theorem 3.2] works in the setting of Lemma 3.2. Alternatively, the existence of the isomorphism $\Theta$ can be seen from the description of the variety $\mathbf{h}^\ast_{\text{reg}^\ast - W'}/W' \times_{\mathbf{h}/\mathbf{h}} \mathbf{h}$ given in Subsection 3.3. Moreover, the isomorphism in Lemma 3.2 can be further restricted as follows. Recall that we have set $\mathcal{E} := \{x \in \mathbf{h} : W_x = W'\}$, which is a closed subvariety inside of $\mathbf{h}^\ast_{\text{reg}^\ast - W'}/W'$. We can form the completion along the closed subvariety $\mathcal{E}$, and we define the algebras $\mathbf{H}_{\mathcal{E}}^{\cdot \mathbf{L}_{\text{reg}^\ast - W'}} := \mathbb{C}[\mathbf{h}^\ast_{\text{reg}^\ast - W'}/W' \wedge L \mathcal{E}] \otimes \mathbb{C}[\mathbf{h}/W'] \mathbf{H}$, $\mathbf{H}_{\mathcal{E}}^{\cdot \mathbf{L}_{\text{reg}^\ast - W'}} := \mathbb{C}[\mathbf{h}^\ast_{\text{reg}^\ast - W'}/W' \wedge L \mathcal{E}] \otimes \mathbb{C}[\mathbf{h}/W'] \mathbf{H}$. The isomorphism $\Theta$ in Lemma 3.2 can be restricted to an isomorphism

$$
\Theta : \mathbf{H}_{\mathcal{E}}^{\cdot \mathbf{L}_{\text{reg}^\ast - W'}} \cong Z(W, W', \mathbf{H}_{\mathcal{E}}^{\cdot \mathbf{L}_{\text{reg}^\ast - W'}}). 
$$

Now let $\hat{\mathcal{E}} \subset \mathbf{E}$ denote the formal neighborhood of $\mathcal{E}$ in $\mathbf{h}^\ast_{\text{reg}^\ast - W'}/W'$, and $\hat{\mathcal{E}}^\times := \hat{\mathcal{E}} \setminus \mathcal{E}$. The algebra of functions on $\hat{\mathcal{E}}^\times$ is a localization of $\mathbb{C}[\mathbf{h}^\ast_{\text{reg}^\ast - W'}/W' \wedge \mathcal{L}]$, and we can form the algebras $\mathbf{H}_{\mathcal{E}}^{\cdot \mathbf{L}_{\text{reg}^\ast - W'}} = \mathbb{C}[\mathbf{h}^\ast_{\text{reg}^\ast - W'}/W' \wedge L \mathcal{E}]$, respectively. The isomorphism $\Theta$ can be further restricted to

$$
\Theta_W : \mathbf{H}_{\mathcal{E}}^{\cdot \mathbf{L}_{\text{reg}^\ast - W'}} \cong Z(W, W', \mathbf{H}_{\mathcal{E}}^{\cdot \mathbf{L}_{\text{reg}^\ast - W'}}). 
$$

**3.3. Supports.** Let us remark that there is a $W$-equivariant isomorphism

$$
\mathbf{h}^\ast_{\text{reg}^\ast - W'}/W' \times_{\mathbf{h}/W} \mathbf{h} \cong \bigsqcup_{w \in W/W'}\mathbf{w}^\ast_{\text{reg}^\ast - W'} \subseteq W/W' \times \mathbf{h}
$$

that, for $x \in \mathbf{h}^\ast_{\text{reg}^\ast - W'}$ and $w \in W$, sends $(W'x, wx)$ to $(wW', wz) \in W/W' \times \mathbf{h}$. Let us denote by $X$ the variety $\bigsqcup_{w \in W/W'}\mathbf{w}^\ast_{\text{reg}^\ast - W'}$. We can think of $\mathbf{H}_{\text{reg}^\ast - W'}$ as $\mathbf{H}(W', X)$, the rational Cherednik algebra associated to the action of $W$ on the variety $X$, see e.g. [Wi, Section 2] for generalities on rational Cherednik algebras associated to the action of a complex reflection group on a smooth algebraic variety (not necessarily a vector space), in this paper we will only work with varieties that are disjoint unions of Zariski open sets inside a vector space. Note that $X \times \mathbf{h}^\ast = T^\ast X$ is a symplectic algebraic variety, and the action of $W$ on $T^\ast X$ is by symplectomorphisms. So $(T^\ast X)/W$ is a Poisson variety. Moreover, it follows from the isomorphism (4) that $X/W = \mathbf{h}^\ast_{\text{reg}^\ast - W'}/W'$ and $(T^\ast X)/W = (T^\ast \mathbf{h}^\ast_{\text{reg}^\ast - W'}/W')$. As before, for a HC $\mathbf{H}_{\text{reg}^\ast - W'}/W'$-bimodule $B$, its support $SS(B) \subseteq (T^\ast \mathbf{h}^\ast_{\text{reg}^\ast - W'}/W')$ is a union of symplectic leaves.

We can describe the symplectic leaves inside $(T^\ast \mathbf{h}^\ast_{\text{reg}^\ast - W'}/W')$ using the results in [BrGo, 7.4]. We remark that [BrGo] works with actions on a vector space, but the proofs work in our setting. Namely, let $W'' \subset W'$ be a parabolic subgroup. Let $\mathcal{L}_{W''}^W := \pi_{\mathbf{h}^\ast_{\text{reg}^\ast - W'}/W'}(\{x \in T^\ast \mathbf{h}^\ast_{\text{reg}^\ast - W'}/W'_x = W''\})$, where $\pi_{\mathbf{h}^\ast_{\text{reg}^\ast - W'}/W'} : T^\ast \mathbf{h}^\ast_{\text{reg}^\ast - W'}/W' \to (T^\ast \mathbf{h}^\ast_{\text{reg}^\ast - W'}/W')/W'$ is the quotient by the $W'$-action. Note that $\mathcal{L}_{W''}^W$ depends only on the conjugacy class of $W''$ in $W'$. The symplectic leaves in $(T^\ast \mathbf{h}^\ast_{\text{reg}^\ast - W'}/W')$ are precisely the $\mathcal{L}_{W''}^W$, where $W'' \subset W'$ is a parabolic
Lemma 3.7. Let $B$ be a HC $\mathbb{H}_{reg-W'}$-bimodule. Assume that $SS(B)$ is a finitely generated Poisson submodule. Consider $B/cB$ as a $\mathbb{C}[\mathcal{X}/W] \otimes \mathbb{C}[h^*/W']$-module where, recall, $\mathcal{X} = \bigcup_{\nu \in W/W'} W_{\nu}^{reg-W'}$, and $\mathcal{X}/W = h^{reg-W'}/W'$. Then, for every nonzero $v \in B/cB$, its support $X_v \subseteq h^{reg-W'}/W' \times h^*/W'$ contains $h^{reg-W'} \times (h^*)^{W'}$.

In view of the description of the symplectic leaves inside $(\mathcal{X} \times h^*)/W'$, Lemma 3.3 is a consequence of the following result.

Lemma 3.4. Let $A$ be a commutative, Noetherian Poisson algebra, and let $M$ be a finitely generated Poisson $A$-module. Then, for every element $m \in M$, its set-theoretic support $X_m \subseteq \text{Spec}(A)$ is a Poisson subvariety.

Proof. First of all, let $I \subseteq A$ be any ideal. For $k \geq 0$, let $M_I := \{n \in M : I^kn = 0\}$. Note that $M_I \subseteq M_{I^{k+1}}$, so that $M(I) := \bigcup_{k \geq 0} M_I$ is a submodule of $M$. We claim that it is a Poisson subvariety. Take $m \in M_I$ and $a \in A$. Let $a_1, \ldots, a_2k \in I$, so that $a_1 \cdots a_{2k}m = 0$. It follows that $0 = \{a_1, a_2, \ldots, a_{2k}m\} = a_1 \cdots a_{2k}(a, m) + \{a_1, a_2, \ldots, a_{2k}\}m$. Thanks to the Leibniz identity again, $\{a_1, a_2, \ldots, a_{2k}\} = a_1 \cdots a_{2k} \in \mathfrak{g}$ and $\mathfrak{g}$ is a Poisson subvariety. Now specialize to the case where $I = \text{Ann}_A(m)$. Since $A$ is Noetherian and $M$ is finitely generated, $M(I) = M_I$ for some $k > 0$. Then, $I^k \subseteq \text{Ann}(M(I))$. On the other hand, since $m \in M(I)$ and $I = \text{Ann}_A(m)$, $\text{Ann}(M(I)) \subseteq I$. So $\sqrt{\text{Ann}(M(I))} = \sqrt{I}$ and the result follows.

Let us remark that, thanks to the correspondence between supports of HC $H_{reg-W'}$- and $H_{reg-W'}$-bimodules, we get from Lemma 3.3 the following result.

Corollary 3.5. Let $B$ be a HC $H_{reg-W'}$-bimodule. Consider $B/cB$ as a $\mathbb{C}[h^{reg-W'}/W'] \otimes \mathbb{C}[h^*/W']$-module. Then, for every nonzero $v \in B/cB$, its support $X_v \subseteq h^{reg-W'}/W' \times h^*/W'$ contains $h^{reg-W'} \times (h^*)^{W'}$.

3.4. Annihilators. We will describe the annihilator of a HC $H_{reg-W'}$-bimodule as a left $\mathbb{C}[h^{reg-W'}/W']$-module. In order to do so, we need the following finiteness result.

Lemma 3.6. Let $B$ be a HC $H_{reg-W'}$-bimodule. Then, $B$ is finitely generated over $\mathbb{C}[h^{reg-W'}/W'][c] \otimes S(c) \mathbb{C}[h^*/W'][c]^{opp}$. Similarly, $B$ is finitely generated over $\mathbb{C}[h^*/W'][c] \otimes S(c) \mathbb{C}[h^{reg-W'}/W'][c]^{opp}$, where the superscript $opp$ means that the corresponding algebra acts on the right.

Proof. Since $B$ is HC, we have that $B/cB$ is a module over $\mathbb{C}[\mathcal{X} \times h^*]/W'$, which is the central algebra $H_{reg-W'}/cH_{reg-W'}$. This latter algebra is finite over its center, so $B/cB$ is a finitely generated module over $\mathbb{C}[\mathcal{X} \times h^*]/W$. Now, the natural map $(\mathcal{X} \times h^*)/W \rightarrow \mathcal{X}/W \times h^*/W'$ is finite, so $B/cB$ is a finitely generated module over $\mathbb{C}[\mathcal{X}]^{W} \otimes \mathbb{C}[h^*]/W'$. Let $z_1, \ldots, z_n$ be generators of $B/cB$ under the action of $\mathbb{C}[\mathcal{X}]^{W} \otimes \mathbb{C}[h^*]/W'$. We can assume that these elements are homogeneous with respect to the grading on $B/cB$ inherited from the one on $B$. Let $v_1, \ldots, v_m$ be homogeneous lifts of $z_1, \ldots, z_n$. It is now standard to see that $v_1, \ldots, v_m$ are generators of $B$ under the action of $\mathbb{C}[\mathcal{X}]^{W} \otimes S(c) \mathbb{C}[h^*]/W'[c]^{opp}$.

Lemma 3.7. Let $B$ be a HC $H_{reg-W'}$-bimodule. Assume that $SS(B) = L^{W'}/W'$. Then, as a (left or right) $\mathbb{C}[h^{reg-W'}/W']$-module, $B$ is annihilated by a power of the ideal $I$ of functions vanishing on $L \subseteq h^{reg-W'}/W'$ where, recall, $L = \{x \in h : W_x = W'\}$.
Proof. First, we show that any element in \( B \) is annihilated by a large enough power of \( I \). Recall that \( \mathcal{L}_{W'} = \mathfrak{h}_{reg}^{W'} \times (\mathfrak{k}^*)^{W'} \). In particular, \( \mathfrak{h}_{reg}^{W'} \times 0 \subset \mathcal{L}_{W'}^{reg} \). It follows by our assumption on \( SS(B) \) that \( I^n \subset \mathbb{C}[\mathfrak{h}^{reg-W'/W}] \subset \mathbb{C}[\mathfrak{h}^{reg-W'/W}] \otimes \mathbb{C}[\mathfrak{k}^{*}] \subset \mathbb{C} [X \times \mathfrak{k}^*]^W \) annihilates \( B/cB \) for \( n \gg 0 \). So for any \( i \in \mathbb{Z} \), \( I_i B \subset \mathfrak{d}^{B^{-1}} \). Now the claim follows because the grading on \( B \) is bounded below.

Now let \( v_1, \ldots, v_m \) be generators of \( B \) as a \( \mathbb{C}[\mathfrak{h}^{reg-W'/W}] |_{c \otimes \mathcal{O}} \mathbb{C}[\mathfrak{k}^{*}]^{\text{opp. module}}, \) and let \( N \gg 0 \) be such that \( I^N v_i = 0 \) for all \( i = 1, \ldots, m \). It is easy to see that \( I^N B = 0 \). We are done. \( \square \)

Let us discuss some consequences of Lemma 3.7. Recall that we have a natural action of the group \( N_{W'}(W) \) on the algebra \( H_{reg-W'} \) by algebra automorphisms, in such a way that the action of \( W' \subset N_{W'}(W) \) coincides with the adjoint action. Recall also that we denote \( \Xi = N_{W'(W'')} \). The map \( \eta_{W'} : \mathfrak{h}^{reg-W'}/W' \to \mathfrak{h}/W \) is étale and it restricts to a covering \( \eta_{W'} : \mathcal{L} \to \eta_{W'}(\mathcal{L}) = \mathcal{L}^{\Xi} \) with Galois group \( \Xi \). This implies that the formal neighborhood \( (\mathfrak{h}/W)^{\eta_{W'}(\mathcal{L})} \) may be identified with the quotient by the action of \( \Xi \) on the formal neighborhood \( (\mathfrak{h}^{reg-W'}/W')^{\Sigma} \). Now let \( B \) be a \( \Xi \)-equivariant HC \( H_{reg-W'} \)-bimodule supported on \( \mathcal{L}_{W'}^{reg} \).

Thanks to Lemma 3.7, \( B \) may be thought of as a quasi-coherent sheaf on an infinitesimal neighborhood of \( \mathcal{L} \subset \mathfrak{h}^{reg-W'}/W' \). Thus, the space of invariants \( B^{\Xi} \) is a quasi-coherent sheaf on an infinitesimal neighborhood of \( \eta_{W'}(\mathcal{L}) \subset \mathfrak{h}/W \), and we may think of it as a quasi-coherent sheaf on \( \mathfrak{h}^{reg-W'}/W' \).

We claim that, moreover, \( B = \mathbb{C}[\mathfrak{h}^{reg-W'}/W'] \otimes_{\mathbb{C}[\mathfrak{h}/W]} \mathfrak{B}^{\Xi} \). Since \( \eta_{W'} \) is, in particular, étale when restricted to \( \mathcal{L} \), the induced map \( (\mathfrak{h}^{reg-W'}/W')^{\Sigma} \to (W'(\mathfrak{h}^{reg-W'}/W')^{\eta_{W'}(\mathcal{L})} \) is the quotient by the free \( \Xi \)-action on the formal neighborhood of \( \mathcal{L} \), and \( \mathbb{C}[\mathfrak{h}^{reg-W'}/W']^{\eta_{W'}(\mathcal{L})} \) may be identified with the algebra of \( \Xi \) invariants in \( \mathbb{C}[\mathfrak{h}^{reg-W'}/W']^{\Sigma} \). The desired equality will follow if we show that the right-hand side is equal to \( \mathbb{C}[\mathfrak{h}^{reg-W'}/W']^{\Sigma} \otimes_{\mathbb{C}[\mathfrak{h}^{reg-W'}/W']^{\eta_{W'}(\mathcal{L})}} B^{\Xi} \). But this is clear by our description of the annihilator of \( B \) (and of \( B^{\Xi} \)).

### 3.5. Main result.

We are now ready to state our main result. Let \( W \) be a parabolic subgroup of \( W \), and let \( B \) be a \( \Xi = N_{W}(W)/W \)-equivariant HC \( H_{reg-W} \)-bimodule. We require that \( SS(B) = \mathcal{L}_{W}^{reg} \), the minimal symplectic leaf in \( \mathfrak{h}^{reg-W}/W \). Recall that we denote \( \eta_{W} : \mathfrak{h}^{reg-W}/W \to \mathfrak{h}/W \). From the previous subsection it follows that \( \eta_{W}(B^{\Xi}) \) is an \( H \)-bimodule satisfying \( (\eta_{W}(B^{\Xi}))_{reg-W} = B \). However, \( \eta_{W}(B^{\Xi}) \) need not be finitely generated over \( H \) so it is not, in general, a HC \( H \)-bimodule. Similarly, if \( W' \) is a parabolic subgroup of \( W \) containing \( W \), \( (\eta_{W}(B^{\Xi}))_{reg-W'} \) does not need to be a HC \( H_{reg-W'} \)-bimodule. However, we can further localize to the punctured formal neighborhood \( \widehat{\mathcal{L}}_{W}^{\times} \). The bimodule \( (\eta_{W}(B^{\Xi}))_{\mathcal{L}^{\times}} \) is now a HC \( H_{\mathcal{L}^{\times}} \)-bimodule.

**Theorem 3.8.** Let \( B \) be a \( \Xi \)-equivariant HC \( H_{reg-W} \)-bimodule. Assume that \( SS(B) = \mathcal{L}_{W}^{reg} \) and that for all parabolic subgroups \( W' \subset W \) in corank 1, there is a HC \( H_{\mathcal{L}_{W}^{\times}} \)-bimodule \( B_{W'} \) whose localization to \( \widehat{\mathcal{L}}_{W}^{\times} \) concides with \( (\eta_{W}(B^{\Xi}))_{\mathcal{L}^{\times}} \). Then, there exists a HC \( H \)-bimodule \( \overline{B} \) such that \( B = \overline{B}_{reg-W} \).

The proof of Theorem 3.8 is inspired by [2, Section 3], where a similar result is shown at the level of category \( \mathcal{O} \) (for the stratum corresponding to the dense symplectic leaf.) The strategy is as follows. We will define a bimodule that is coherent over an open subset in \( \mathfrak{h}/W \) whose complement has codimension 2. Then, we can take global sections. This open subset will be the image in \( \mathfrak{h}/W \) of

\[
\mathfrak{h}^{sr-W} := \bigcup_{W \subseteq W'} \mathfrak{h}^{reg-W'}.
\]

It is clear that the complement of \( \mathfrak{h}^{sr-W} \) in \( \mathfrak{h} \) has codimension 2. Moreover, \( \mathfrak{h}^{W} \cap \mathfrak{h}^{sr-W} \) is an open subset of \( \mathfrak{h}^{W} \) whose complement has codimension at least 2. Indeed, we have

\[
\mathfrak{h}^{W} \cap \mathfrak{h}^{sr-W} = \mathfrak{h}^{W} \setminus \bigcup_{s,s' \in W \cap s,s' \notin \Gamma_{s} \cap \Gamma_{s'}} \Gamma_{s} \cap \Gamma_{s'}.
\]

(5)
The way to get a desired bimodule is as follows. First, for each parabolic subgroup $W'$ containing $W$ in corank 1, we will construct a HC $H_{reg-W'}$-bimodule with the property that its lift to $\mathfrak{h}^{reg-W}/W$ coincides with $B$. Then we will get our bimodule by, roughly speaking, glueing the bimodules defined over $\mathfrak{h}^{reg-W}/W'$.

**Proof of Theorem 3.8.** Part 1: Constructing HC $H_{reg-W'}$-bimodules. For each parabolic subgroup $W'$ containing $W$ in corank 1, let $B_{W'}$ be a HC $H_{L_{W'}}$-bimodule that localizes to $(\eta_{W'}(B))^\times$. Note that we may assume that $B_{W'} \subseteq (\eta_{W'}(B)^\times)_{L_{W'}}$, if this is not the case we can just replace $B_{W'}$ by its quotient by the maximal sub-bimodule that is killed by the localization, we can find such a sub-bimodule because $B_{W'}$ is a finitely generated bimodule over the noetherian algebra $H_{L_{W'}}$.

On the other hand, let $\eta_{W'} : \mathfrak{h}^{reg-W'}/W' \to \mathfrak{h}/W$ be the natural projection, and consider $\eta_{W'}(\eta_{W'}(B)) = \mathbb{C}[\mathfrak{h}^{reg-W'}/W'] \otimes_{\mathbb{C}[\mathfrak{h}/W]} B^\times$. The inclusion $\mathbb{C}[\mathfrak{h}^{reg-W'}/W'] \subseteq \mathbb{C}[\hat{L}_{W'}]$ induces a map $\eta_{W'}(\eta_{W'}(B)) \to \eta_{W'}(B)_{\hat{L}_{W'}}$ that we claim to be injective. Indeed, this follows because inside $(\mathfrak{h} \oplus *)/W$ we have $L_{W'} \subseteq \hat{L}_{W'}$ and the singular support of any finitely generated $H$-sub-bimodule of $\eta_{W'}(B)$ (which is the union of its HC sub-bimodules) contains $L_{W'}$. The claim is now a consequence of the fact that $L_{W'}$ is the minimal symplectic leaf inside $T^*\chi_{W'}/W$, cf. Subsection 3.3.

Define $\tilde{B}_{W'} := (\eta_{W'}(\eta_{W'}(B)))_{\hat{L}_{W'}}$. Note that this is an $H_{reg-W'}$-bimodule. Let us see that it is finitely generated. By a suitable straightforward adaptation of Lemma 3.6, $(\eta_{W'}(B))_{\hat{L}_{W'}}$ is finitely generated over the algebra $\mathbb{C}[\hat{L}_{W'}][c] \otimes_{S(\hat{C}[\mathfrak{h}])} \mathbb{C}[\mathfrak{h}][c]^\text{opp}$. Note that $B_{W'}$ is a $\mathbb{C}[\hat{L}_{W'}][c] \otimes_{S(\hat{C}[\mathfrak{h}])} \mathbb{C}[\mathfrak{h}][c]^\text{opp}$-lattice inside of $(\eta_{W'}(B))_{\hat{L}_{W'}}$. So what we need to show is that $\eta_{W'}(\eta_{W'}(B)) \cap L$ is finitely generated over $\mathbb{C}[\mathfrak{h}^{reg-W'}/W'][c] \otimes_{S(\hat{C}[\mathfrak{h}])} \mathbb{C}[\mathfrak{h}][c]^\text{opp}$ for some lattice $L$. We can produce such a lattice as follows. Again thanks to Lemma 3.6 $B$ is finitely generated over $\mathbb{C}[\mathfrak{h}^{reg-W'}/W'][c] \otimes_{S(\hat{C}[\mathfrak{h}])} \mathbb{C}[\mathfrak{h}][c]^\text{opp}$, so we have an epimorphism $\Upsilon : (\mathbb{C}[\mathfrak{h}^{reg-W'}/W'][c] \otimes_{S(\hat{C}[\mathfrak{h}])} \mathbb{C}[\mathfrak{h}][c]^\text{opp})^{\oplus n} \to B$, which in turn induces an epimorphism $\bar{\Upsilon} : (\mathbb{C}[\hat{L}_{W'}][c] \otimes_{S(\hat{C}[\mathfrak{h}])} \mathbb{C}[\mathfrak{h}][c]^\text{opp})^{\oplus n} \to (\eta_{W'}(B))_{\hat{L}_{W'}}$. We take $L$ to be the image of the restriction of $\bar{\Upsilon}$ to $(\mathbb{C}[\hat{L}_{W'}][c] \otimes_{S(\hat{C}[\mathfrak{h}])} \mathbb{C}[\mathfrak{h}][c]^\text{opp})^{\oplus n}$. This is clearly a lattice. Since $\mathbb{C}[\mathfrak{h}^{reg-W'}/W'] \cap \mathbb{C}[\hat{L}_{W'}] = \mathbb{C}[\mathfrak{h}^{reg-W'}/W']$ we have that $L \cap \eta_{W'}(\eta_{W'}(B))$ coincides with the intersection of $\eta_{W'}(\eta_{W'}(B))$ with the image of the restriction of $\bar{\Upsilon}$ to $(\mathbb{C}[\mathfrak{h}^{reg-W'}/W'][c] \otimes_{S(\hat{C}[\mathfrak{h}])} \mathbb{C}[\mathfrak{h}][c]^\text{opp})^{\oplus n}$. So $L \cap \eta_{W'}(\eta_{W'}(B))$ is finitely generated over $\mathbb{C}[\mathfrak{h}^{reg-W'}/W'][c] \otimes_{S(\hat{C}[\mathfrak{h}])} \mathbb{C}[\mathfrak{h}][c]^\text{opp}$. Note that it follows that $\tilde{B}_{W'}$ is a HC $H_{reg-W'}$-bimodule with $SS(\tilde{B}_{W'}) = \tilde{B}_{W'}$.

It remains to show that $\tilde{B}_{W'}^{reg-W} = B$. Since $B = (\eta_{W'}(B))^{reg-W} = (\eta_{W'}(B))^{reg-W}$ it is enough to check that the lift of $\tilde{B}_{W'}$ to $\mathfrak{h}^{reg-W'}/W'$ coincides with that of $(\eta_{W'}(B))^{reg-W'}$. By definition, $\tilde{B}_{W'} \subseteq (\eta_{W'}(B))^{reg-W'} \subseteq (\eta_{W'}(B))_{\hat{L}_{W'}}$. Now, for every $b \in (\eta_{W'}(B))_{\hat{L}_{W'}}$ there exists $f \in \mathbb{C}[\hat{L}_{W'}]$ vanishing on $\hat{L}_{W'}$ with $fb \in B_{W'}$. If, moreover, $b \in (\eta_{W'}(B))^{reg-W'}$, then $f \in \mathbb{C}[\mathfrak{h}^{reg-W'}/W']$ and $fb \in \tilde{B}_{W'}$. This implies the desired result.

Part 2: Glueing. First we will define a sheaf on $\mathfrak{h}$, then we will take $W$-invariants to pass to $\mathfrak{h}/W$. For each parabolic subgroup $W'$ containing $W$ in corank 1, let $\pi_{W'} : \mathfrak{h}^{reg-W'} \to \mathfrak{h}^{reg-W'}/W'$ be the projection, and $\iota_{W'} : \mathfrak{h}^{reg-W'} \to \mathfrak{h}$ the inclusion. So we can consider $\iota_{W'}^*\pi_{W'}^*\tilde{B}_{W'}$. We will take the intersection of these sheaves, so we need to find a sheaf containing all of them. Since $\mathfrak{h}^{reg-W'} \subseteq \mathfrak{h}^{sr-W}$ for all $W'$, this will be a sheaf defined on $\mathfrak{h}^{sr-W}$. So let $\pi : \mathfrak{h} \to \mathfrak{h}/W$ and $\iota : \mathfrak{h}^{sr-W} \to \mathfrak{h}$ be the natural projection and inclusion, respectively. By the construction, viewing $\eta_{W'}(B)$ as a quasicoherent sheaf on $\mathfrak{h}/W$, we may think of $\iota_{W'}^*\pi_{W'}^*\tilde{B}_{W'}$ as being contained inside of $\iota_{W}^*\pi_{W}^*\eta_{W'}(B)$. So the intersection

$$\tilde{B} := \bigcap_{W \subseteq W'} \iota_{W}^*\pi_{W}^*\eta_{W'}(B)$$

makes sense and is a sheaf on $\mathfrak{h}^{sr-W}$.
Note that $W$ acts naturally on $\iota_*\pi^*\eta_{W^*}(B^\Xi)$. Now notice that, for a parabolic subgroup $W' \subseteq W$ and $w \in W$, we have a canonical, graded isomorphism $\mathcal{H}_{reg-W'} \cong \mathcal{H}_{reg-wW'/w-1}$. Indeed, recall that $\mathcal{H}_{reg-W}$ is the rational Cherednik algebra for the action of $W$ on $\mathcal{X}_{W'}$, a disjoint union of Zariski open subsets of $\mathfrak{h}$, cf. Subsection 3.3. It is clear that $\mathcal{X}_{W'} = \mathcal{X}_{wW'/w-1}$ and the isomorphism between the algebras follows. So tracing back the construction, we see that we can pick our bimodules $B_{\mathcal{L}_{W'}}$ in such a way that, for $w \in W$, $w(\iota_*\pi^*\eta_{W^*}B_{W'}) = \iota_{wW'/w-1}*\pi^*\eta_{W'/w-1}B_{wW'/w-1}$. So $B$ is $W$-stable. Finally, define

$$\widehat{B} := (\pi, B)^W,$$

where $\pi : \mathfrak{h} \to \mathfrak{h}/W$ is the projection. We claim that $\widehat{B}$ is stable under the bimodule action of $\mathcal{H}$. To see this first note that, by definition, $\widehat{B} = \pi_*\widehat{B} \cap J_*B^\Xi$, where $J : \mathfrak{h}^{reg-W} \to \mathfrak{h}/W$ is the projection. Each one of the bimodules on the right-hand side of the previous equality is stable under the (left or right) action of $\mathcal{H}$. So $\widehat{B}$ is also $\mathcal{H}$-stable.

Now set $\mathcal{B} := \Gamma((\pi(\mathfrak{h}^{reg-W}), \widehat{B}))$. We have that $\mathcal{B}$ is a $\mathcal{H}$-bimodule. We claim that it is HC. First of all, since $\mathcal{B} \subseteq J_*B^\Xi$, we have that $\mathcal{B}$ is $C[\mathfrak{h}]$-flat. It is also clear that $\mathcal{B}/\mathfrak{h}\mathcal{B}$ is a module over $Z(\mathcal{H}/\mathfrak{h}\mathcal{H})$ and that $\mathcal{B}$ is graded. So, to finish the claim that $\mathcal{B}$ is HC, we need to show that it is finitely generated. We will show that, in fact, $\mathcal{B}/\mathfrak{h}\mathcal{B}$ is finitely generated over the algebra $C[\mathfrak{h}/W] \otimes C[\mathfrak{h}^{reg-W}]$. The following is an easy consequence of [Wi] Lemma 3.6.

**Lemma 3.9.** Let $X$ be an affine Noetherian scheme, and let $U \subseteq X$ be an open subset of $X$ whose complement has codimension at least 2. Let $M$ be a coherent sheaf on $U$, and assume that the support of any global section $m \in \Gamma(U, M)$ contains an irreducible component of $U$. Then, $\Gamma(U, M)$ is finitely generated over $C[X]$.

**Proof.** By [Wi] Lemma 3.6, we get that $\Gamma(U, M)/\Gamma(U, M)$ is a finitely generated $C[X]/IC[X]$-module, where $I$ is the nilradical of $C[X]$. The result follows. 

Note that we can look at $\mathcal{B}/\mathfrak{c}\mathcal{B}$ as a coherent sheaf on an infinitesimal neighborhood $U$ of $\pi(\mathfrak{h}^{reg-W} \cap \mathfrak{h}/W) \times \mathfrak{h}^{reg-W}/W$, this follows from our assumptions on the singular support of $\mathcal{B}$, $SS(\mathcal{B}) = \mathcal{L}_{W_1}^W$, the construction of $\mathcal{B} \subseteq J_*\mathcal{B}$ and Lemma 3.7. This infinitesimal neighborhood may be regarded as an open set inside an infinitesimal neighborhood $X$ of $\pi(\mathfrak{h}^{reg-W}) \times \mathfrak{h}^{reg-W}/W$. Now, it follows from Lemma 3.3 that the support of any global section $m \in \Gamma(U, \mathcal{B}/\mathfrak{c}\mathcal{B})$ contains $\pi(\mathfrak{h}^{reg-W} \cap \mathfrak{h}/W) \times \mathfrak{h}^{reg-W}/W$. Then, it follows from Lemma 3.9 that $\mathcal{B}/\mathfrak{c}\mathcal{B}$ is finitely generated over $C[X]$. In particular, it is finitely generated over $C[\mathfrak{h}/W] \otimes C[\mathfrak{h}^{reg-W}]$, this follows because the codimension of the complement of $\mathfrak{h}^{reg-W}$ in $\mathfrak{h}$ is 2. Then, $\mathcal{B}$ is a HC $\mathcal{H}$-bimodule. By construction, $\mathcal{B}_{reg-W} = \mathcal{B}$. This finishes the proof of Theorem 3.8. 

Let us remark one important feature of the bimodule $\mathcal{B}$ we have constructed: it has no sub-bimodules whose singular support is properly contained inside $\mathcal{L}_{W_1}^W$. Indeed, this follows from Corollary 3.5 and the fact that $\mathcal{B} \subseteq \mathcal{B}^\Xi$.

### 3.6. Specializing parameters.

Let $c, c' : C[\mathcal{S}]^W \to C$ be parameters. Recall that $R_h(H_C)$, $R_h(H_{c'})$ are quotients of $\mathcal{H}$ and that, if $B$ is a HC $H_{c'}H_{c'}$-bimodule with a good filtration, then $R_h(B)$ is a HC $\mathcal{H}$-bimodule. A similar result holds for HC $H_{c',reg-W}H_{c,reg-W}$-bimodules. We remark that, since $\mathfrak{h}^{reg-W}$ is in degree 0, the Rees construction commutes with localization: for an affine, open subset $U \subseteq \mathfrak{h}/W$, $R_h(B)|_U = R_h(B)|_U$. We also remark that the Bezrukavnikov-Etingof isomorphisms hold in the specialized setting. So we can take a $\Xi$-equivariant HC $H_{c,reg-W}H_{c',reg-W}$-bimodule $B$ such that $SS(B) = \mathcal{L}_{W_1}^W$ and for every parabolic subgroup $W'$ containing $W$ in corank 1, the $H_c(W', \mathfrak{h})_{\mathcal{L}_{W_1}^W}$ - $H_{c'}(W', \mathfrak{h})_{\mathcal{L}_{W_1}^W}$-bimodule $B^\Xi_{\mathcal{L}_{W_1}^W}$ is the localization of a HC $H_c(W', \mathfrak{h})_{\mathcal{L}_{W_1}^W}$ - $H_{c'}(W', \mathfrak{h})_{\mathcal{L}_{W_1}^W}$-bimodule. By Theorem 3.8 we can find a HC $\mathcal{H}$-bimodule $\mathcal{B}$ that lifts to $R_h(B)$. Since $\mathcal{B}_{\mathcal{L}_{W_1}^W}$ is a bimodule over $R_h(H_c(W', \mathfrak{h})_{\mathcal{L}_{W_1}^W})R_h(H_{c'}(W', \mathfrak{h})_{\mathcal{L}_{W_1}^W})$, we see that the bimodule $\mathcal{B}$ factors through $R_h(H_c)-R_h(H_{c'})$, so $\mathcal{B}/(h-1)\mathcal{B}$ is a HC $H_cH_{c'}$-bimodule that lifts to $B$. We summarize this discussion in the following theorem, which is a specialized version of Theorem 3.8.
Theorem 3.10. Let \( W \) be a parabolic subgroup of \( W \), and let \( B \) be a \( \Xi \)-equivariant HC \( H_{c,\mathrm{reg}} \)-\( H_{c',\mathrm{reg}} \)-bimodule, where \( c, c' \in \mathbb{C}[S] \) are conjugation invariant functions. Assume that \( \text{SS}(B) = \mathcal{L}_{/W}' \) and that for all minimal parabolic subgroups \( W' \subseteq W \) containing \( W \), the bimodule \( (\eta_{/W}(B\Xi))_{/W'} \) is the localization of a HC \( H_{c}(W', h)_{/W'} - H_{c'}(W', h)_{/W'} \)-bimodule. Then, there exists a HC \( H_{c}H_{c'} \)-bimodule \( \overline{B} \) such that \( \overline{B}_{\text{reg} - W} = B \).

We will use Theorem 3.10 to find bimodules in the image of the restriction functor \( \bullet_{/W} \). Let us be explicit about how we are going to achieve this.

Corollary 3.11. Let \( W \) be a parabolic subgroup of \( W \), and let \( B \) be a finite dimensional \( \Xi \)-equivariant \( H_{c}H_{c'} \)-bimodule. Assume that for every parabolic subgroup \( W' \) containing \( W \) in corank 1 there exists a HC \( H_{c}(W') - H_{c'}(W') \)-bimodule \( B_{W'} \) with \( (B_{W'})_{/W'} = B \), where the \( N_{W'}(W)/W \)-equivariant structure on \( B \) is restricted from that of \( \Xi \). Then, there exists a HC \( H_{c}(W) - H_{c'}(W) \)-bimodule \( \overline{B} \) with \( \overline{B}_{/W} = B \).

Proof. Recall the functor \( \mathcal{G} \) defined in Subsection 2.1. Since \( B \) is finite dimensional, \( \text{SS}(\mathcal{G}(R_{\mathcal{h}}(B))) = \mathcal{L}_{/W}' \). Our assumptions on \( B \) imply, since \( \mathcal{G} \) is a fully faithful embedding, that \( \mathcal{G}(R_{\mathcal{h}}(B)) \) satisfies the conditions of Theorem 3.10. So we can find a HC \( \mathcal{H} \)-bimodule \( \mathcal{B} \) with \( \mathcal{B}_{\text{reg} - W} = \mathcal{G}(R_{\mathcal{h}}(B)) \). Note that since \( \text{SS}(\mathcal{G}(R_{\mathcal{h}}(B))) = \mathcal{L}_{/W}' \), \( (\mathcal{B}_{\text{reg} - W})_{\Xi} = \mathcal{B}_{\text{reg} - W} \). Thus, by the construction of the restriction functor, \( \mathcal{B}_{/W} = R_{\mathcal{h}}B \). It remains to put \( \mathcal{B} := \mathcal{B}/(\mathcal{h} - 1) \mathcal{B} \).

4. Localization of HC bimodules.

The previous section, with \( W = \{1\} \), gives a necessary and sufficient condition for a \( \mathcal{D}(\mathfrak{h}^{\text{reg}})\# W \)-bimodule to be the localization of a HC \( H_{c}H_{c'} \)-bimodule. In this section, we will see which bimodules can appear as the localization of irreducible HC bimodules. This will allow us to see, in particular, for which irreducible modules \( M \in \mathcal{O} \), the space of locally finite maps \( \text{Hom}_{\text{fin}}(\mathcal{A}_{\mathfrak{c}}(\text{triv}), M) \) can be nonzero. In the next section, we will use this result to give a classification of irreducible HC bimodules with full support for rational Cherednik algebras associated to Coxeter groups.

We would like to remark that, after writing a preliminary version of this paper, we found out that most of the results in this section are already contained in some form in [Sp]. There it is assumed that all parameters are regular (i.e., that \( \mathcal{O} \) is a semisimple category or, equivalently, that the algebra \( H_{c} \) is simple) but, under minor modifications we provide here, the results are still valid in the general case, see Remark 4.9.

4.1. Bimodules over \( \mathcal{D}(\mathfrak{h}^{\text{reg}})\# W \). Recall that, independently of the parameter \( c, H_{c,\text{reg} / W} \) is isomorphic to \( \mathcal{D}(\mathfrak{h}^{\text{reg}})\# W \). So we start by studying bimodules over the latter algebra. Since \( W \) acts freely on \( \mathfrak{h}^{\text{reg}} \), the algebras \( \mathcal{D}(\mathfrak{h}^{\text{reg}})\# W \) and \( \mathcal{D}(\mathfrak{h}^{\text{reg}}/W) \) are Morita equivalent, an equivalence is given by \( M \mapsto e M \) where \( e = |W|^{-1} \sum_{w \in W} w \) is the trivial idempotent for \( W \). Throughout this section, we denote \( X := \mathfrak{h}^{\text{reg}}/W \).

We will relate HC \( H_{c} \)-bimodules to spaces of differential operators on local systems on \( X \). So, first, we recall Grothendieck’s definition of differential operators: if \( M \) and \( N \) are \( \mathbb{C}[X] \)-modules, then the space of \( \mathbb{C}[X] \)-differential operators is a subspace of \( \text{Hom}_{\mathbb{C}}(M, N) \), defined via an increasing filtration \( \text{Diff}(M, N) = \bigcup_{n \geq 0} \text{Diff}(M, N)_{n} \), where the components \( \text{Diff}(M, N)_{n} \) are inductively defined as follows:

\[
\text{Diff}(M, N)_{-1} := 0, \quad \text{Diff}(M, N)_{n+1} := \{ f \in \text{Hom}_{\mathbb{C}}(M, N) : [a, f] \in \text{Diff}(M, N)_{n} \text{ for all } a \in \mathbb{C}[X] \}.
\]

If \( M, N \) are \( \mathcal{D}(X) \)-modules, then \( \text{Diff}(M, N) \) is a \( \mathcal{D}(X) \)-bimodule. We remark that if \( N \) is a local system then we have a \( \mathcal{D}(X) \)-bimodule isomorphism, \( N \otimes_{\mathbb{C}[X]} \mathcal{D}(X) \cong \text{Diff}(\mathbb{C}[X], N) \), where the flat connection on \( N \otimes_{\mathbb{C}[X]} \mathcal{D}(X) \) is as in [HTT, Proposition 1.2.9]. An explicit isomorphism is given by \( n \otimes_{\mathbb{C}[X]} d \mapsto (f \mapsto d(f)n) \). Note that this implies that \( \text{Diff}(\mathbb{C}[X], N) \) is finitely generated both as a right and as a left \( \mathcal{D}(X) \)-module whenever \( N \) is a local system. As a right \( \mathcal{D} \)-module, an explicit set of generators is \( n_{1} \otimes_{\mathbb{C}[X]} 1, \ldots, n_{i} \otimes_{\mathbb{C}[X]} 1 \), where \( n_{1}, \ldots, n_{i} \) are generators of the \( \mathbb{C}[X] \)-module \( N \). This set also generates \( N \otimes_{\mathbb{C}[X]} \mathcal{D}(X) \) as a left \( \mathcal{D} \)-module. Since the algebra \( \mathcal{D}(X) \) is simple and noetherian, [Br, Theorem 10] implies the following.

Lemma 4.1. Let \( N \) be a nonzero local system. Then, \( \text{Diff}(\mathbb{C}[X], N) \) is a pregenerator in both categories of left and right \( \mathcal{D}(X) \)-modules. In particular, \( \text{Diff}(\mathbb{C}[X], N) \otimes_{\mathcal{D}(X)} M \neq 0 \) (resp. \( M \otimes_{\mathcal{D}(X)} \text{Diff}(\mathbb{C}[X], N) \neq 0 \)) for any nonzero left (resp. right) \( \mathcal{D}(X) \)-module \( M \).
Now assume that $N$ is an irreducible local system. We have a natural evaluation map of $\mathcal{D}(X)$-modules
\[ \text{Diff}(\mathbb{C}[X], N) \otimes_{\mathcal{D}(X)} \mathbb{C}[X] \rightarrow N, \phi \otimes f \mapsto \phi(f). \]
Since $\text{Diff}(\mathbb{C}[X], N) \neq 0$, the evaluation map is not zero. Then, by the simplicity of $N$, this map is surjective. We claim that it is also injective. To see this, note that we have an isomorphism $\text{Diff}(\mathbb{C}[X], N) \otimes_{\mathcal{D}(X)} \mathbb{C}[X] \cong N$ of $\mathbb{C}[X]$-modules. This follows from the description of $\text{Diff}(\mathbb{C}[X], N)$ above. So we can view the evaluation map as an element of $\text{End}_{\mathbb{C}[X]}(N, N)$. Now $N$ is noetherian since it is finitely generated over $\mathbb{C}[X]$. We have seen that the evaluation map is surjective. Hence, it must also be injective. This discussion has the following consequence.

**Proposition 4.2.** If $N$ is an irreducible local system over $X$, then the $\mathcal{D}(X)$-bimodule $\text{Diff}(\mathbb{C}[X], N)$ is irreducible.

**Proof.** Assume $\text{Diff}(\mathbb{C}[X], N)$ has a nontrivial sub-bimodule $V$, and let $V'$ be the quotient bimodule. The result of [Br] Theorem 10 implies that any nonzero sub-bimodule or quotient module of $\text{Diff}(\mathbb{C}[X], N)$ must be a progenerator of $\mathcal{D}(X)$-mod and mod-$\mathcal{D}(X)$. Then, $\text{Tor}_{\mathcal{D}(X)}^1(V', N) = 0$ and we have a short exact sequence
\[ 0 \rightarrow V \otimes_{\mathcal{D}(X)} N \rightarrow \text{Diff}(\mathbb{C}[X], N) \otimes_{\mathcal{D}(X)} N \rightarrow V' \otimes_{\mathcal{D}(X)} N \rightarrow 0. \]
By the discussion above, the module $\text{Diff}(\mathbb{C}[X], N) \otimes_{\mathcal{D}(X)} N$ is irreducible. This forces one of $V \otimes_{\mathcal{D}(X)} N$ or $V' \otimes_{\mathcal{D}(X)} N$ to be $0$. But both $V, V'$ are progenerators of $\mathcal{D}(X)$-mod and mod-$\mathcal{D}(X)$. This is a contradiction. □

4.2. Localization of HC bimodules. We will apply the results of the previous subsection to give an explicit description of the localization of certain Harish-Chandra $H_c-H_c'$-bimodules. Namely, consider the Verma module $\Delta_c'(\text{triv})$. We will consider bimodules of the form $\text{Hom}_{\mathcal{H}_\mathfrak{t}\mathcal{A}_\mathfrak{t}}(\Delta_c'(\text{triv}), N)$, for $N \in \mathcal{O}_c$ with full support. We first see that any irreducible HC $H_c-H_c'$-bimodule with full support is contained in such a bimodule.

**Proposition 4.3.** Let $V$ be an irreducible $H_c-H_c'$-bimodule with full support. Then, there exists an irreducible object $T \in \mathcal{O}_c$ such that $V$ is a sub-bimodule of $\text{Hom}_{\mathcal{H}_\mathfrak{t}\mathcal{A}_\mathfrak{t}}(\Delta_c'(\text{triv}), T)$.

**Proof.** Since $V$ has full support, we see that $V[\delta^{-1}] \neq 0$, so $e(V[\delta^{-1}])e \neq 0$. Note that $e(V[\delta^{-1}])e$ is a bimodule over the algebra $\mathcal{D}(X)$ that is noetherian as either a left or right $\mathcal{D}(X)$-module. Then, it is a progenerator of $\mathcal{D}(X)$-mod, so $e(V[\delta^{-1}])e \otimes_{\mathcal{D}(X)} \mathbb{C}[X] \neq 0$. In particular, this implies that $V \otimes_{\mathcal{D}(X)} \Delta_c'(\text{triv}) \neq 0$. Let $T$ be an irreducible quotient of the latter module. It is easy to see that we have a nonzero bimodule map $V \rightarrow \text{Hom}_{\mathcal{H}_\mathfrak{t}\mathcal{A}_\mathfrak{t}}(\Delta_c'(\text{triv}), T)$. □

Note that for an irreducible module $N \in \mathcal{O}_c$, we have that $e\text{Hom}_{\mathcal{H}_\mathfrak{t}\mathcal{A}_\mathfrak{t}}(\Delta_c'(\text{triv}), N)[\delta^{-1}]e$ is a $\mathcal{D}(X)$-bimodule. We claim that this bimodule is isomorphic to $\text{Diff}(\mathbb{C}[X], N_X)$ whenever the former bimodule is nonzero and $N_X := eN[\delta^{-1}]$. This claim follows from the following result.

**Lemma 4.4.** Let $N \in \mathcal{O}_c$ be an irreducible module with full support. For any standard module $\Delta_c'(\tau)$, the bimodule $e\text{Hom}_{\mathcal{H}_\mathfrak{t}\mathcal{A}_\mathfrak{t}}(\Delta_c'(\tau), N)[\delta^{-1}]e$ is isomorphic to a sub-bimodule of $\text{Diff}(e\Delta_c'(\tau)[\delta^{-1}], eN[\delta^{-1}])$.

**Proof.** Set $M := \Delta_c'(\tau)$. Let $f \in \text{Hom}_{\mathcal{H}_\mathfrak{t}\mathcal{A}_\mathfrak{t}}(M, N)$. Since for some $m$, $\delta^m$ is $W$-invariant, we have that $(ad(e\delta^m))^k f = 0$, so for every $x \in M$,
\[ \sum_{j=0}^{k} (-1)^j \binom{k}{j} (e\delta^m)^{(k-j)} f((e\delta^m)^j x) = 0. \]
Then, since $M$ is free as a $\mathbb{C}[\mathfrak{h}]$-module we can extend $f$ to $eM[\delta^{-1}]$ by
\[ f(\delta^{-m} x) = -(e\delta^m)^{-k} \sum_{j=1}^{k} (-1)^j \binom{k}{j} (e\delta^m)^{(k-j)} f((e\delta^m)^{(j-1)} x). \]
To see that this actually defines an inclusion, assume that $f \neq 0$. Then, $f(x) \neq 0$ for some element $x \in M$. Since $N$ is torsion-free (see e.g. [GGOR Proposition 5.21]), the element $f(x)$ is not a zero divisor. This implies that the image of $f$ in $\text{Diff}(e\Delta_c'(\tau)[\delta^{-1}], eN[\delta^{-1}])$ is nonzero. □
Corollary 4.5. Let $B$ be an irreducible HC $H_c$-$H_{c^\prime}$-bimodule with full support. Then, there exists an irreducible local system $N$ on $X$ such that $eB[\delta^{-1}]e = \text{Diff}(\mathbb{C}[X], N)$.

Proof. By Proposition 4.3 and Lemma 4.4, we have that $eB[\delta^{-1}]e \subseteq \text{Diff}(\mathbb{C}[X], N)$ for some irreducible local system $N$. But $\text{Diff}(\mathbb{C}[X], N)$ is an irreducible $\mathcal{D}(X)$ bimodule, Proposition 4.2. We are done. \qed

Corollary 4.6. We have an isomorphism $H_c \cong \text{Hom}_{\mathfrak{f}_{1n}}(\Delta_c(\text{triv}), \Delta_c(\text{triv}))$.

Proof. Reasoning as in the proof of Lemma 4.4, we have that $\text{Hom}_{\mathfrak{f}_{1n}}(\Delta_c(\text{triv}), \Delta_c(\text{triv}))$ is a HC $H_c$-bimodule whose localization to $\mathfrak{h}^{\text{reg}}$ is an irreducible $\mathcal{D}(\mathfrak{h}^{\text{reg}})\#W$-bimodule. So $\text{Hom}_{\mathfrak{f}_{1n}}(\Delta_c(\text{triv}), \Delta_c(\text{triv}))$ contains a unique irreducible bimodule with full support. It is easy to see that any subbimodule of $\text{Hom}_{\mathfrak{f}_{1n}}(\Delta_c(\text{triv}), \Delta_c(\text{triv}))$ has full support, so this bimodule has an irreducible socle. In particular, it is indecomposable.

On the other hand, we have a natural map $H_c \to \text{Hom}_{\mathfrak{f}_{1n}}(\Delta_c(\text{triv}), \Delta_c(\text{triv}))$, $x \mapsto (m \mapsto x m)$. Since the representation $\Delta_c(\text{triv})$ is faithful, this is an inclusion. Then, by Proposition 2.10, we have that $H_c$ must be isomorphic to a direct summand of $\text{Hom}_{\mathfrak{f}_{1n}}(\Delta_c(\text{triv}), \Delta_c(\text{triv}))$. By the previous paragraph, we must have $H_c \cong \text{Hom}_{\mathfrak{f}_{1n}}(\Delta_c(\text{triv}), \Delta_c(\text{triv}))$. \qed

Remark 4.7. We remark that the isomorphism in Corollary 4.6 is also an algebra isomorphism with respect to the composition structure on $\text{Hom}_{\mathfrak{f}_{1n}}(\Delta_c(\text{triv}), \Delta_c(\text{triv}))$. This generalizes [BEG, Proposition 8.10 (i)].

4.3. KZ functor. Since for any HC $H_c$-$H_{c^\prime}$-bimodule $B$ and any module $M \in \mathcal{O}_{c^\prime}$, the module $B \otimes_{H_{c^\prime}} M$ is a module in category $\mathcal{O}_c$, it makes sense to ask what is the image of a module of the form $B \otimes_{H_{c^\prime}} M$ under the KZ functor. In this subsection, we answer this question when $M$ has the form $\text{Hom}_{\mathfrak{f}_{1n}}(\Delta_c(\text{triv}), M)$ for an irreducible module with full support $M \in \mathcal{O}_c$. Namely, we have the following result.

Lemma 4.8. Let $c, c^\prime : S \to \mathbb{C}$ be conjugation invariant functions and consider the rational Cherednik algebras $H_c, H_{c^\prime}$. Let $q, q^\prime$ be the associated sets of parameters for the Hecke algebras $H_q, H_{q^\prime}$, so that we have $KZ_c : \mathcal{O}_c \to H_q$-mod, $KZ_{c^\prime} : \mathcal{O}_{c^\prime} \to H_{q^\prime}$-mod. Let $M \in \mathcal{O}_c$ be an irreducible module with full support. Assume that $\text{Hom}_{\mathfrak{f}_{1n}}(\Delta_c(\text{triv}), M) \neq 0$. Then, for every finite dimensional module $N \in H_{q^\prime}$-mod, the $\pi_1(X)$-module $KZ_c(M) \otimes_{\mathbb{C}} N$ factors through $H_{q^\prime}$.

Proof. We show that for every $\tilde{N} \in \mathcal{O}_{c^\prime}$:

$$KZ_{c^\prime}(\text{Hom}_{\mathfrak{f}_{1n}}(\Delta_c(\text{triv}), M) \otimes_{H_{c^\prime}} \tilde{N}) = KZ_c(M) \otimes_{\mathbb{C}} KZ_{c^\prime}(\tilde{N}).$$

Since $H_{q^\prime}$-mod is a quotient of $\mathcal{O}_{c^\prime}$ via the KZ functor, this implies the result. Now, by results in the previous subsection the localization to $X$ of $\text{Hom}_{\mathfrak{f}_{1n}}(\Delta_c(\text{triv}), M) \otimes_{H_{c^\prime}} N$ is isomorphic to $\text{Diff}(\mathbb{C}[X], M_X) \otimes_{\mathcal{D}(X)} N_X$. Since $M_X$ is a local system, $\text{Diff}(\mathbb{C}[X], M_X) \cong M_X \otimes_{\mathbb{C}[X]} \mathcal{D}(X)$. Then,

$$(\text{Hom}_{\mathfrak{f}_{1n}}(\Delta_c(\text{triv}), M) \otimes_{H_{c^\prime}} \tilde{N})_X = M_X \otimes_{\mathbb{C}[X]} \mathcal{D}(X) \otimes_{\mathcal{D}(X)} \tilde{N}_X = M_X \otimes_{\mathbb{C}[X]} \tilde{N}_X.$$

By [HTT] Proposition 4.7.8, $DR(M_X \otimes_{\mathbb{C}[X]} \tilde{N}_X)$ is precisely $DR(M_X) \otimes_{\mathbb{C}} DR(\tilde{N}_X)$, with diagonal action of the braid group $\pi_1(X)$. The lemma is proved. \qed

Remark 4.9. Note that $\Delta_c(\text{triv})$ has an irreducible socle. This follows because $\Delta_c(\text{triv})$ is free as a $\mathbb{C}[\mathfrak{h}]$-module (so that every submodule is torsion free) and the localization $\Delta_c(\text{triv})^{\mathfrak{h}^{\text{reg}}}$ is an irreducible $\mathcal{D}(\mathfrak{h}^{\text{reg}})\#W$-module. Moreover, for $S := \text{Soc}(\Delta_c(\text{triv}))$, we get $KZ(S) = KZ(\Delta_c(\text{triv}))$. Then, Lemma 4.8 holds, with the same proof, if we substitute $\Delta_c(\text{triv})$ by $S$. We will mostly use this form of the lemma.

5. HC bimodules with full support.

5.1. Semisimplicity. Recall that we denote by $\overline{\mathcal{HC}}(c, c^\prime)$ the quotient category of all HC $H_c$-$H_{c^\prime}$-bimodules by the full subcategory consisting of those bimodules whose support is properly contained in $(\mathfrak{h} \oplus \mathfrak{h}^*)/W$. In this subsection, we show a result that was promised in Subsection 2.9, namely, that the category $\overline{\mathcal{HC}}(c, c^\prime)$ is semisimple for different parameters $c, c^\prime$. The following preparatory result is key for the results in this section.

Lemma 5.1. Assume that the category $\overline{\mathcal{HC}}(c, c^\prime)$ is nonzero. Then, there exists a 1-dimensional character $\tau$ of $W$ such that $\text{Hom}_{\mathfrak{f}_{1n}}(S, \Delta_c(\tau)) \neq 0$ where, recall $S \in \mathcal{O}_c$ is the irreducible module that gets sent to the trivial representation under the KZ functor.
Proof. Assume that \( \overline{\text{HC}}(c,c') \neq 0 \). Then, there exists an irreducible module \( N \in \mathcal{O}_c \) with full support such that \( \text{Hom}_{\mathfrak{H}_c}(S,N) \neq 0 \). Now, consider the pointwise stabilizer \( W \subseteq W^r \). This is a cyclic group. Note that, since \( \text{Hom}_{\mathfrak{H}_c}(S,N) \neq 0 \), we have that \( \text{Hom}_{\mathfrak{H}_c}(\text{Res}_{W}^{W^r}(S),\text{Res}_{W}^{W^r}(N)) \neq 0 \). This follows from Lemma 2.11. Since KZ commutes with restriction, we have that \( S_{W^r} \) is the unique subquotient of \( \text{Res}_{W}^{W^r}(S) \) that will full support, which implies that \( \text{Hom}_{\mathfrak{H}_c}(S_{W^r},\text{Res}_{W}^{W^r}(N)) \neq 0 \). Now, in category \( O \) for the rational Cherednik algebra of \( W^r \), for every irreducible representation (= 1-dimensional character) \( \tau \) of \( W^r \), we have that either \( L_{c}(\tau) = \Delta_{c}(\tau) \) or \( L_{c}(\tau) \) is finite dimensional. Since \( \text{Res}_{W}^{W^r}(N) \) has full support, we conclude that there exists an irreducible representation \( \tau_{\Gamma} \) of \( W^r \) with \( \text{Hom}_{\mathfrak{H}_c}(S_{W^r},\Delta_{c}(\tau_{\Gamma})) \neq 0 \). We remark that we can take \( \tau_{\Gamma} = \tau_{\Gamma'} \) if \( \Gamma, \Gamma' \in \mathcal{A} \) are conjugate, this follows from the conjugation invariance of \( c \).

Now recall that we have an isomorphism \( \text{Hom}(W,\mathbb{C}^X) \rightarrow \left( \prod_{\Gamma \in \mathcal{A}} \text{Hom}(W_{\Gamma},\mathbb{C}^X) \right) / W \) that is given by restriction, cf. [R 3.3.1]. So the \( W \)-equivariant choice of characters \( \{ \tau_{\Gamma} : \Gamma \in \mathcal{A} \} \) determines a 1-dimensional character \( \tau \) of \( W \). We claim that \( \text{Hom}_{\mathfrak{H}_c}(\Delta_{c}(\tau)) \neq 0 \). To see this, we will use Corollary 3.11. Assume for the moment that for every reflection hyperplane \( \Gamma \in \mathcal{A} \), \( \text{Hom}_{\mathfrak{H}_c}(W_{\Gamma},\Delta_{c}(\tau)) \neq 0 \). Then, \( \text{Hom}_{\mathfrak{H}_c}(S_{W^r},\text{Res}_{W}^{W^r}(\Delta_{c}(\tau)))_{W^r} \) is a nonzero submodule of the 1-dimensional bimodule \( \text{Hom}_{C}(\mathbb{C},KZ_{c}(\Delta_{c}(\tau))) \), so the conditions of Corollary 3.11 are satisfied. Using this lemma we get a \( \text{HC} H_{c}\)-bimodule \( B \) that localizes to \( \text{Hom}_{C}(KZ_{c}(\Delta_{c}(\tau))) \). Using Lemma 4.8 it is easy to see that, up to subquotients with proper support, \( B = \text{Hom}_{\mathfrak{H}_c}(S_{W^r},\Delta_{c}(\tau)) \). So what we need to show now is that \( \text{Hom}_{\mathfrak{H}_c}(S_{W^r},\text{Res}_{W}^{W^r}(\Delta_{c}(\tau))) \neq 0 \) for every \( \Gamma \in \mathcal{A} \). We proceed to do this.

The \( H_{c}(W_{\Gamma}) \)-module \( \text{Res}_{W}^{W^r}(\Delta_{c}(\tau)) \) has a standard filtration, see e.g. [Sh, Proposition 1.9]. Since, by construction, the restriction of the representation \( \tau \) to \( W_{\Gamma} \) is \( \tau_{\Gamma} \), Proposition 3.14(ii) in [BR], allows us to conclude that \( \text{Res}_{W}^{W^r}(\Delta_{c}(\tau)) = \Delta_{c}(\tau_{\Gamma}) \). So \( \text{Hom}_{\mathfrak{H}_c}(S_{W^r},\text{Res}_{W}^{W^r}(\Delta_{c}(\tau))) \neq 0 \). We are done.

\[ \text{Corollary 5.2.} \quad \text{Assume that} \quad \overline{\text{HC}}(c,c') \neq 0. \quad \text{Then, the categories} \quad \overline{\text{HC}}(c,c') \quad \text{and} \quad \overline{\text{HC}}(c',c') \quad \text{are equivalent. Moreover, they are equivalent to the category of representations of} \quad W/W' \quad \text{for some normal subgroup} \quad W' \quad \text{of} \quad W. \]

\[ \text{Proof.} \quad \text{Let} \quad B \quad \text{be a} \quad \text{HC} H_{c}\text{-bimodule with full support. By the previous lemma, we may assume that} \quad B = \text{Hom}_{\mathfrak{H}_c}(S_{\Delta_{c}(\tau)}) \quad \text{for a 1-dimensional character} \quad \tau \quad \text{of} \quad W, \quad \text{so that} \quad eB[\delta^{-1}]e = \text{Diff}(\mathbb{C}[X],N). \quad \text{Here,} \quad N = c\Delta_{c}(\tau)[\delta^{-1}] \quad \text{is a rank 1 local system. Then, the tensor product functor} \quad eB[\delta^{-1}]e \otimes \mathcal{D}(X) \quad \text{induces a self-equivalence in the category of} \quad \mathcal{D}(X)\text{-bimodules. Indeed, this follows because} \quad eB[\delta^{-1}]e = N \otimes_{\mathbb{C}[X]} \mathcal{D}(X) \quad \text{and} \quad N \quad \text{is a line bundle on} \quad X. \quad \text{This implies that} \quad B \otimes_{H_{c}} \mathcal{D}(X) : \text{HC}(c',c') \rightarrow \text{HC}(c,c') \quad \text{induces an equivalence between} \quad \overline{\text{HC}}(c',c') \quad \text{and} \quad \overline{\text{HC}}(c,c'). \quad \text{The last assertion was checked in Subsection 2.9}. \]

5.2. Subgroup \( W_{c} \). Recall that the category \( \overline{\text{HC}}(c,c) \) is equivalent to the category of representations of \( W/W' \) for some normal subgroup \( W' \subseteq W \). Here, we describe the group \( W' \). To motivate our description, we first look at the case where \( W \) is a cyclic group.

So assume \( W = \mathbb{Z}/\ell\mathbb{Z} \), with generator \( s \). The Hecke algebra \( \mathcal{H}_{q} \) is the quotient of the polynomial algebra \( \mathbb{C}[T] \) by the ideal generated by the polynomial \( (T - 1) \prod_{i=1}^{\ell - 1} (T - q_{i}) \). We remark that \( q_{i} \) is the scalar by which \( T \) acts on \( KZ_{c}(\mathbb{C}) \), where \( \mathcal{C} \) is the irreducible representation of \( W \) where \( s \) acts by multiplication by \( \exp(2\pi i \ell^{-1}/\ell) \). Now, if \( \text{Hom}_{\mathfrak{H}_c}(\Delta(\text{triv}),\Delta(\mathcal{C})) \) is nonzero then, thanks to Lemma 4.8, multiplication by \( q_{i} \) induces a map \( q \rightarrow q_{i} \) where \( q \) denotes the multiset \( q = \{ q_{0} = 1, q_{1}, \ldots, q_{\ell - 1} \} \). It is not hard to see that this map is actually a bijection, i.e. it preserves multiplicities. In particular, \( q_{i} \) is an \( \ell \)-root of 1.

So set \( \eta := \exp(2\pi i \ell^{-1}/\ell) \). Note that the group \( W \) acts on the set of Hecke parameters, the element \( s^{k} \) acts on a multiset \( q' = \{ q_{0}', \ldots, q'_{\ell - 1} \} \) by multiplying each element by \( \eta^{k} \). The stabilizer of \( q \), the Hecke parameter associated to the Cherednik parameter \( c_{i} \), is cyclic, so it is generated by \( s^{m} \), where \( m \) divides \( \ell \), say \( mp = \ell \). By definition, \( W_{c} := \{ s^{m} \} \). Note that for generic \( c \) we have that \( m = \ell \), so \( W_{c} = W \).

Let us generalize the definition of \( W_{c} \) for the case where \( W \) is any complex reflection group. Fix a reflection hyperplane \( \Gamma \in \mathcal{A} \). Let \( \eta_{\Gamma} := \exp(2\pi i \ell^{-1}/\ell) \). Now consider the set \( \mathcal{X}_{\Gamma} := \{ i \in \{ 1, \ldots, \ell_{\Gamma} \} : \eta_{\Gamma}^{j} q_{\Gamma,j} \in \{ q_{\Gamma,0} = 1, \ldots, q_{\Gamma,\ell_{\Gamma} - 1} \} \} \) with the same multiplicity as \( q_{\Gamma,j} \) for every \( j = 0, \ldots, \ell_{\Gamma} - 1 \). For example, \( \ell_{\Gamma} = X_{\Gamma} \). Now set \( \eta := \min \mathcal{X}_{\Gamma} \). It is clear that \( \eta \) is a divisor of \( \ell_{\Gamma} \), say \( \eta \mid \ell_{\Gamma} \). We define
$W_c := \langle s_{\Gamma}^{p_{\Gamma}} : \Gamma \in \mathcal{A} \rangle \subseteq W$. By definition, this is a reflection group. Note that the conjugation invariance of $c$ implies that $W_c$ is a normal subgroup of $W$.

Note that $W_c = \{1\}$ if and only if $m_{\Gamma} = 1$ for every reflection hyperplane $\Gamma \in \mathcal{A}$. This happens if and only if \{q_{\Gamma,0}, q_{\Gamma,1}, \ldots, q_{\Gamma,\ell_{\Gamma} - 1}\} = \{\eta_1, \eta_2, \ldots, \eta_{\ell_{\Gamma} - 1}\}$, that is, if and only if $c \in p_Z$. On the other hand $W_c = W$ if and only if $m_{\Gamma} = \ell_{\Gamma}$ for every $\Gamma \in \mathcal{A}$, and this is a generic condition. Finally, it is clear that $W_c = W_c'$ provided $c' \in p_Z$.

**Theorem 5.3.** The category $\overline{HC}(c, c)$ is equivalent to the category of representations of $W/W_c$.

To prove Theorem 5.3 we will check that in this case there exists a parameter $c' \in p_Z + c$ such that the algebra $H_c(W)$ decomposes as $W \#_{W_c} H_c(W_c)$, for some parameter $c \in \mathbb{C}[S \cap Wc][W]$ which is naturally computed from $c'$. Since $c - c' \in p_Z$, the categories $\overline{HC}(c, c)$ and $\overline{HC}(c', c')$ are equivalent. The result will now follow if we check that $H_{W_c}(W_c)$ has a unique irreducible HC bimodule with full support.

**5.3. Proof of Theorem 5.3**. We continue to use the notation introduced in Subsection 5.2.

Assume, for the moment, that the parameter $c$ is such that, for $\Gamma \in \mathcal{A}$, $c(s_{\Gamma}^{i}) = 0$ unless $i = p_{\Gamma}, 2p_{\Gamma}, \ldots, (m_{\Gamma} - 1)p_{\Gamma}$. Then, it is clear from the relations $[1]$ that the $H_c$-subalgebra generated by $b_{\Gamma} b_{\Gamma}^{*}$ and $W_c$ is isomorphic to $H_c(W_c)$, where $c$ simply denotes the restriction of the parameter $c$ to $W_c$. So $H_c$ is generated by $H_c(W_c)$ and $W_c$. Moreover, the subalgebra $H_c(W_c)$ is stable under the adjoint action of $W$. It follows that $H_c \cong H_c(W_c) \#_{W_c} W$, where the latter algebra is $H_c(W_c) \otimes^{\mathbb{C}} W$ with product defined analogously to the smash-product algebra, using the action of $W$ on $H_c(W_c)$. Thus, $HC$-bimodules with full support correspond to $W$-equivariant HC $H_{W_c}(W_c)$-bimodules with full support, where the action of $W_c \subseteq W$ coincides with that coming from the inclusion $W_c \subseteq H_c(W_c)$.

Let us now examine the Hecke parameters $q_{\Gamma,i}$, still under the assumption that $c(s_{\Gamma}^{i}) = 0$ unless $i = p_{\Gamma}, 2p_{\Gamma}, \ldots, (m_{\Gamma} - 1)p_{\Gamma}$. It follows easily from $[2]$ that $k_{\Gamma,i} = k_{\Gamma,i + m_{\Gamma}}$ for all $i$. But then it follows that:

$$q_{\Gamma,i + m_{\Gamma}} = \exp(2\pi \sqrt{-1}(k_{\Gamma,i} - i - m_{\Gamma}/\ell_{\Gamma})) = \eta_{\Gamma}^{-m_{\Gamma}} q_{\Gamma,i}$$

Note that, given numbers $Q_{\Gamma,0} = 1, Q_{\Gamma,1}, \ldots, Q_{\Gamma,m_{\Gamma} - 1} \in \mathbb{C}^*$ we can always find a parameter $c \in \mathbb{C}[S][W]$ with $c(s_{\Gamma}^{i}) = 0$ unless $i$ is a multiple of $p_{\Gamma}$ and such that $q_{\Gamma,i} = Q_{\Gamma,i}$. This implies the following.

**Lemma 5.4.** Let $c \in \mathbb{C}[S][W]$ be a parameter of the form considered in Subsection 5.2. Then, there exists a parameter $c' \in \mathbb{C}[S][W]$ such that for every $\Gamma \in \mathcal{A}$, $c(s_{\Gamma}^{i}) = 0$ unless $i = p_{\Gamma}, 2p_{\Gamma}, \ldots, (m_{\Gamma} - 1)p_{\Gamma}$ and $H_q = H_{\Gamma}$, that is, $c - c' \in p_Z$.

**Lemma 5.5.** Assume that $c - c' \in p_Z$. Then, the categories $\overline{HC}(c, c)$ and $\overline{HC}(c', c')$ are equivalent.

**Proof.** Let $\chi$ be a character of $W$. It is enough to show that the categories $\overline{HC}(c, c)$ and $\overline{HC}(c + \overline{\chi}, c + \overline{\chi})$ are equivalent, see Subsection 2.4. Recall that we have an isomorphism $e H_{c + \overline{\chi}} \cong e_{\chi} H_{c + \overline{\chi}}$, so that $e_{H_{c + \overline{\chi}}} \otimes e_{H_{c + \overline{\chi}}} \otimes e_{H_{c + \overline{\chi}}}$ is a $e H_{c + \overline{\chi}} \otimes e_{H_{c + \overline{\chi}}} \otimes e_{H_{c + \overline{\chi}}}$-bimodule. Moreover, this bimodule is HC, see e.g. [L4, Lemma 3.2]. We remark that, by construction, it has full support. So we get a $HC$ $H_{c + \overline{\chi}}$-bimodule with full support, namely $B_{c + \overline{\chi}} := H_{c + \overline{\chi}} \otimes e_{H_{c + \overline{\chi}}} \otimes e_{H_{c + \overline{\chi}}} H_{c + \overline{\chi}}$. Thanks to Corollary 5.2 we have that the categories $\overline{HC}(c, c)$ and $\overline{HC}(c + \overline{\chi}, c + \overline{\chi})$ are equivalent. By a similar argument, now using that $\overline{HC}(c, c + \overline{\chi}) \neq 0$, the categories $\overline{HC}(c + \overline{\chi}, c)$ and $\overline{HC}(c + \overline{\chi}, c + \overline{\chi})$ are equivalent. We are done.

**Proof of Theorem 5.3.** Thanks to Lemmas 5.4 and 5.5, we may assume that $c(s_{\Gamma}^{i}) = 0$ unless $i$ is a multiple of $p_{\Gamma}$, i.e. that $H_{c} \cong H_{c}(W_c) \#_{W_c} W$. We claim now that $H_{c}(W_c)$ has a unique irreducible HC bimodule with full support. For $\Gamma \in \mathcal{A}$, let $q_{\Gamma,0} = 1, q_{\Gamma,1}, \ldots, q_{\Gamma,m_{\Gamma} - 1}$ be the parameters for the Hecke algebra $H_{c}^\prime(W_c)$ associated to $c$, and denote $\eta_{\Gamma} := \exp(2\pi \sqrt{-1}/m_{\Gamma}) = \eta_{\Gamma}^{p_{\Gamma}}$. We also denote $m'_{\Gamma} := \min\{i \in \{1, \ldots, m_{\Gamma}\} : \eta_{\Gamma}^{i} q_{\Gamma,i} \in q_{\Gamma}$ with the same multiplicity as $q_{\Gamma,i}$ for every $j = 0, \ldots, m_{\Gamma} - 1\}$. Thanks to Lemma 4.8 our claim will follow if we check the following.

**Claim:** For every hyperplane $\Gamma \in \mathcal{A}$, $m'_{\Gamma} = m_{\Gamma}$.

We proceed by contradiction. Assume there exists $0 < i < m_{\Gamma}$ such that, for every $j = 0, \ldots, m_{\Gamma} - 1$, $\eta_{\Gamma}^{i} q_{\Gamma,j}$ is in the multiset $q_{\Gamma,j}$ with the same multiplicity as $q_{\Gamma,j}$. Note that we have
Thus, \( \eta^j_i q_{r,j} \in q_r \) implies that \( \eta^j_i q_{r,j} \in \{ q_{r,0}, \ldots, q_{r,m_r-1} \} \), with the same multiplicity as \( q_{r,j} \). But \( q_r = \{ q_{r,0}, \ldots, q_{r,m_r-1}, \eta^{m_r} q_{r,0}, \ldots, \eta^{m_r} q_{r,m_r-1}, \eta^{(p_r-1)m_r} q_{r,0}, \ldots, \eta^{(p_r-1)m_r} q_{r,m_r-1} \} \). Thus, we see that \( \eta^j_i q_{r,j} \in q_r \) with the same multiplicity as \( q_{r,j} \) for every \( j = 0, \ldots, \ell_r - 1 \). This contradicts the choice of \( m_r \). Thus, \( H_c(W_c) \) has a unique irreducible HC bimodule with full support. Since \( H_c = H_c(W_c) \# W_c W \), this proves Theorem 5.3

5.4. Two-parametric setting. We have given a description of the category \( \Pi C(c, c) \). The goal of this subsection is to extend Theorem 5.3 to a two-parametric setting, that is, to find a description of the category \( \Pi C(c, c') \) for different parameters \( c, c' \). By Corollary 5.2, it is enough to find those pairs of parameters \( (c, c') \) such that \( \Pi C(c, c') \) is nonzero. This is done in the following result.

**Proposition 5.6.** Harish-Chandra \( H_c \)-\( H_{c'} \)-bimodules with full support exist if and only if there exists a character \( \varepsilon : W \to \mathbb{C}^\times \) such that \( \varepsilon c - c' \in p_Z \).

First of all, we remark that, if \( \varepsilon : W \to \mathbb{C}^\times \) is a character, the algebras \( H_c \) and \( H_{\varepsilon c} \) are isomorphic, an isomorphism \( H_c \to H_{\varepsilon c} \) is given by \( x \mapsto x, y \mapsto y, w \mapsto \varepsilon(w)w, x \in h^*, y \in h, w \in W \). This isomorphism preserves the algebras \( \mathbb{C}[h]^W, \mathbb{C}[h]^W \). So, for any two parameters \( c, c' \), the categories \( HC(c, c') \) and \( HC(\varepsilon c, c') \) are equivalent, via an equivalence that preserves the support of a HC bimodule. This clearly implies that, if \( \varepsilon c - c' \in p_Z \), then \( \Pi C(c, c') \neq 0 \).

Now assume \( \Pi C(c, c') \neq 0 \). Thanks to Lemma 5.1 there exists a character \( \varepsilon : W \to \mathbb{C}^\times \) such that \( \text{Hom}_{\Pi C(\varepsilon c)}(\Delta_c(\vtriv), \Delta_c(\varepsilon)) \neq 0 \). Under the equivalence \( \varphi_\ast : O_c \to O_{\varepsilon c} \) coming from the isomorphism \( \varphi : H_c \to H_{\varepsilon c} \), we have that \( \varphi_\ast \Delta_c(\vtriv) = \Delta_{\varepsilon c}(\varepsilon) \). Thus, we have \( \text{Hom}_{\Pi C(\varepsilon c)}(\Delta_c(\vtriv), \Delta_{\varepsilon c}(\vtriv)) \neq 0 \). Then, Proposition 5.6 is a consequence of the following result.

**Lemma 5.7.** Assume \( \text{Hom}_{\Pi C(\varepsilon c)}(\Delta_c(\vtriv), \Delta_{\varepsilon c}(\vtriv)) \neq 0 \). Then, \( c - c' \in p_Z \).

**Proof.** First of all, note that \( c \in p_Z \) if and only if \( c|_{W_g} \in p_Z(W_g) \) for every reflection hyperplane \( \Gamma \in \mathcal{A} \). Since \( \text{Res}_{\Pi C}(\Delta_c(\vtriv)) = \Delta_c(\vtriv_{W_g}) \) where \( ? = c, c' \), we have that \( \text{Hom}_{\Pi C}(\Delta_c(\vtriv), \Delta_{\varepsilon c}(\vtriv)) \neq 0 \) implies \( \text{Hom}_{\Pi C(\varepsilon c)}(\Delta_c(\vtriv_{W_g}), \Delta_{\varepsilon c}(\vtriv_{W_g})) \neq 0 \) for every \( \Gamma \in \mathcal{A} \). Thus, we may assume that \( W \) is a cyclic group.

So assume \( \text{Hom}_{\Pi C}(\Delta_c(\vtriv), \Delta_{\varepsilon c}(\vtriv)) \neq 0 \). Since \( \text{KZ}(\Delta_c(\vtriv)) = \text{C} \), the trivial representation of \( \mathcal{H}_q \), Lemma 4.8 implies that \( \mathcal{H}_q \)-mod \( \subseteq \mathcal{H}_q \)-mod as full subcategories of \( \mathbb{C}[t, t^{-1}] \)-mod. But \( \mathcal{H}_q \), \( \mathcal{H}_q' \) are commutative algebras of the same dimension. This implies that \( \mathcal{H}_q = \mathcal{H}_q' \). Thus, \( c - c' \in p_Z \). This concludes the proof of Lemma 5.7 and of Proposition 5.6.

We remark that the statement Theorem 1.1 simply combines the statements of Corollary 5.2, Theorem 5.3 and Proposition 5.6.

6. Type A.

6.1. Preliminary results. We now turn our attention to type A, that is, \( W = S_n \), with reflection representation \( h = \{ (x_1, \ldots, x_n) \in \mathbb{C}^n : \sum x_i = 0 \} \). Throughout this section, we denote \( H_c(n) := H_c(S_n) \). Similarly, we denote \( H_q(n) := H_q(S_n) \), the Hecke algebra associated to \( S_n \) with parameter \( q \in \mathbb{C}^\times \). In this subsection, we gather some results on the structure of the algebra \( H_c \) and category \( O_c \).

It is known, cf. [BB Example 3.25], [L Theorem 5.8.1], that the algebra \( H_c := H_c(n) \) is simple unless \( c = r/m \) with \( r, m \in \mathbb{Z} \), \( \gcd(r; m) = 1 \) and \( 1 < m \leq n \). In this case, [L Theorem 5.8.1 (2)], the algebra \( H_c \) has \( [n/m] \) proper nonzero two-sided ideals that are linearly ordered by inclusion, say \( J_1 \subset J_2 \subset \cdots \subset J_{[n/m]} \). Moreover, \( J_i^2 = J_i \) for any \( i = 1, \ldots, [n/m] \). We set \( J_0 := \{ 0 \} \), \( J_{[n/m]}+1 := H_c \).

The classification of two-sided ideals gives a characterization of the possible supports of HC bimodules. For \( i = 1, \ldots, [n/m] \) consider the subgroup \( S_{n,i}^\sim \subseteq S_n \), and consider the set \( X_i := \{ x \in h \oplus h^* : W_x = S_{n,i}^\sim \} \). Let \( L_i \) be the image of \( X_i \) under the natural projection \( h \oplus h^* \to (h \oplus h^*)/S_n \). This is a symplectic leaf. The support of \( H_c/J_i \) is \( L_i \). Now recall Lemma 2.4 that says that for a HC \( H_c H_{c'} \)-bimodule \( B \), \( SS(B) = SS(H_c/LAnn(B)) = SS(H_c/RAnn(B)) \). This implies that \( HC(c, c') = 0 \) unless \( c, c' \) have the same

\( q_{r,j} = \exp \left( \frac{2\pi \sqrt{-1} (k_i - i)}{m_r} \right) = q_{r,j}^{p_r} \)
namely, since \( c = r/m \), \( c' = r'/m' \), with \( \gcd(r; m) = \gcd(r'; m') = 1 \) and \( 1 < m \leq n \).

We now give a description of the supports of irreducible modules in \( \mathcal{O}_c \). Namely, for every \( i = 1, \ldots, \lfloor n/m \rfloor \), let \( X'_i = \{ x_1, \ldots, x_n \} \subseteq \mathbb{C}^n : x_1 = 0, x_1 = x_2 = \cdots = x_m, x_{m+1} = \cdots = x_{2m}, \ldots, x_{(i-1)m+1} = \cdots = x_{im} \}, \) and let \( X_i \) be the union of the \( \mathfrak{g}_n \) translates of \( X'_i \), so that \( \mathbb{C}^n \supset \cdots \supset X_1 \supset X_i \). Then, \[ \text{BE} \text{ Example 3.25}, \] \[ \text{Wi} \text{ Theorem 3.9} \] any module in category \( \mathcal{O}_c \) is supported on one of the \( X_i \). Denote by \( \mathcal{O}^i_c \) the full subcategory of \( \mathcal{O}_c \) consisting of all modules whose support is contained in \( X_i \). Note that this is a Serre subcategory of \( \mathcal{O}_c \). Let us explain a description of the category \( \mathcal{O}^i_c/\mathcal{O}^{i+1}_c \) obtained in \[ \text{Wi} \] Theorem 1.8. Let \( p := n - im \), \( q := \exp(2\pi\sqrt{1c}) \) and consider the Hecke algebra \( H_q(p) \). Then \[ \text{Wi} \text{ Theorem 1.8} \] tells us that the category \( \mathcal{O}^i_c/\mathcal{O}^{i+1}_c \) is equivalent to the category of finite dimensional modules over the algebra \( \mathfrak{g}_{c} \otimes H_q(p) \).

Let us recall how \[ \text{Wi} \text{ Theorem 1.8} \] is proved, as this will be important for our arguments. So let \( i \) and \( p \) be as in the previous paragraph. Consider the subgroup \( \mathfrak{g}_{c}^{xi} \subseteq \mathfrak{g}_n \). Let \( h := \{ x \in h : \mathfrak{g}_{c}^{xi} \subseteq \text{Stab}_{\mathfrak{g}_n}(x) \} (= X'_i) \) and \( h^{reg} := \{ x \in h : \text{Stab}_{\mathfrak{g}_n}(x) = \mathfrak{g}_{c}^{xi} \} \). Then, Wilcox proves that we have a localization functor, \( \text{Loc}^i : \mathcal{O}^i \rightarrow \mathcal{D}^{h^{reg}}/(\mathfrak{g}_i \otimes C_p) \)-mod, \( M \mapsto C[h^{reg}] \otimes_{C[h]} M \) that factors through \( \mathcal{O}^i_c/\mathcal{O}^{i+1}_c \) and that identifies this quotient category with a subcategory of the category of \( \mathfrak{g}_{c} \otimes \mathcal{O}_c \)-equivariant \( \mathcal{D}^{h^{reg}} \)-modules with regular singularities. Then, he checks that under the Riemann-Hilbert correspondence that identifies the latter category with the category of finite dimensional representations of \( \pi_1(\mathcal{D}^{h^{reg}}/(\mathfrak{g}_i \otimes \mathcal{O}_c)) \), the image of \( \mathcal{O}^i_c/\mathcal{O}^{i+1}_c \) gets identified with the subcategory \( C[\mathfrak{g}_i \otimes \mathcal{H}_q(p)] \)-mod of \( \pi_1(\mathcal{D}^{h^{reg}}/(\mathfrak{g}_i \otimes \mathcal{O}_c)) \)-rep where, recall, \( q := \exp(2\pi\sqrt{1c}) \). We denote by \( \text{KZ}^i : \mathcal{O}^i \rightarrow (C[\mathfrak{g}_i \otimes \mathcal{H}_q(p)]) \)-mod the composition of the localization functor \( \text{Loc}^i \) with the Riemann-Hilbert correspondence.

This construction has the following consequence for HC bimodules. Let \( S \in \mathcal{O}_c \) be the irreducible module supported on \( X_1 \) that gets sent to the trivial \( \mathcal{O}^i_c \otimes H_q(p) \)-module under \( \text{KZ}^i \) so that, in particular, \( \text{Loc}^i(S) = C[h^{reg}] \). Then, the proofs in Subsection 4.3 can be carried out in this setting and we see that, whenever \( T \) is a simple module with \( \text{Hom}_{\mathfrak{g}_c}(S, T) \neq 0 \), and \( N \) is another simple module in \( \mathcal{O}^i \), then \( \text{KZ}^i(\text{Hom}_{\mathfrak{g}_c}(S, T) \otimes_{H_q} N) = \text{KZ}^i(S) \otimes C \text{KZ}^i(T) \).

**Lemma 6.1.** Let \( S \in \mathcal{O}^i \) be the irreducible module satisfying \( \text{KZ}^i(S) = \mathbb{C} \), the trivial \( \mathfrak{g}_{c} \otimes H_q(p) \)-module. Let \( T \in \mathcal{O}_c \) (necessarily supported on \( X_1 \)) be a simple module satisfying \( \text{Hom}_{\mathfrak{g}_c}(S, T) \neq 0 \). Then, for every \( M \in C[\mathfrak{g}_i \otimes \mathcal{H}_q(p)] \)-mod, the \( \pi_1(\mathcal{D}^{h^{reg}}/(\mathfrak{g}_i \otimes \mathcal{O}_c)) \)-module \( \text{KZ}^i(T) \otimes M \) factors through \( C[\mathfrak{g}_i \otimes \mathcal{H}_q(p)] \).

The previous lemma gives an upper bound on the number of irreducible objects in the category \( \mathcal{H}_{\mathcal{L}_c}(H_c) \), namely, since \( c \not\in \mathbb{Z} \) then, using Lemma 4.8 we see that, if \( T \in \mathcal{O}^i_c \) is such that \( \text{Hom}_{\mathfrak{g}_c}(S, T) \neq 0 \), then \( \text{KZ}^i(T) \) has to be of the form \( \lambda \otimes \mathbb{C} \), where \( \lambda \) is an irreducible representation of \( \mathfrak{g}_i \) and \( \mathbb{C} \) stands for the trivial \( \mathcal{H}_q(p) \) representation.

**Proposition 6.2.** The number of irreducible HC bimodules whose support coincides with the closure of the symplectic leaf \( L_i \) is no more than \( p(i) \), the number of partitions of \( i \).

We will see in Subsection 6.4 that the bound obtained in Proposition 6.2 is sharp.

### 6.2. Semisimplicity of \( \mathcal{H}_{\mathcal{L}_c}(H_c) \).

We have just obtained an upper bound on the number of irreducible objects of the category \( \mathcal{H}_{\mathcal{L}_c}(H_c) \). In this subsection, we check that this category is semisimple. The proof is based on restriction functors, Subsection 2.9. Recall that this is a functor \( \bullet : \mathcal{H}_{\Xi}(H_c) \rightarrow \mathcal{H}_{\Xi}(H_e) \) that identifies the quotient category \( \mathcal{H}_{\mathcal{L}_c}(H_e) \) with a full subcategory of the category of finite dimensional \( \Xi \)-equivariant HC \( H_e \) bimodules that is closed under taking subquotients where, recall, \( \Xi = N_W(W)/W \). Then, we start with a few remarks on finite dimensional bimodules.

Recall that the algebra \( H_c(n) \) has a finite dimensional module if and only if \( c = r/n \), with \( \gcd(r; n) = 1 \), \[ \text{BE} \text{ Theorem 1.2} \]. The unique irreducible finite dimensional \( H_c \)-module is \( L_c(\text{triv}) \) if \( c > 0 \); and it is \( L_c(\text{sign}) \) if \( c < 0 \). Moreover, the category of finite dimensional \( H_c \)-modules is semisimple, this follows either from the results of the previous subsection or from the fact that irreducibles in \( \mathcal{O}_c \) do not admit self-extensions, see e.g. \[ \text{BE} \text{ Proposition 1.12} \]. Note that it follows that \( H_e \) has a unique irreducible finite dimensional bimodule, and that this bimodule does not have self-extensions.

We remark that a finite dimensional bimodule must be HC: given a finite dimensional bimodule \( M \), for any element \( x \in \mathbb{C}[h]^W \cup \mathbb{C}[h^*]^W \), there exists \( n \gg 0 \) such that \( x^n M = 0 = M x^n \). This follows from the existence of a grading on \( M \) as a left \( H_e \)-module compatible with a grading on \( H_e \) given by
may be identified with simple quotients of spherical rational Cherednik algebras. Namely, for positive integers $n,N$ coprime) consider the Cherednik algebras $H^{c}$ at the end of this subsection. The following is our main result.

Proposition 6.4. For convenience, we assume that $c < r/m$, we will deal with the case $c = r/m$, where $m$ divides $n$ and $i = n/m$. It turns out that the case $2m = n$ is important in our argument for the general case, that we explain in Subsection 6.4.

6.3. Bimodules with minimal support. We give a complete description of the category of HC $H_{c}(n)$-bimodules with minimal support when the parameter $c$ has the form $c = r/m$, for $m$ a divisor of $n$. In particular, we show that in this case the normal subgroup $N$ in Proposition 6.3 is trivial. Throughout this subsection, we denote $k := n/m$. For convenience, we assume that $c > 0$, we will deal with the case $c < 0$ at the end of this subsection. The following is our main result.

Proposition 6.4. Let $c := r/m$, where gcd$(r;m) = 1$ and $m$ is a divisor of $n$. Let $k := n/m$. Then, the category $HC_{L}(H_{c}(n))$ of HC $H_{c}$-bimodules with minimal support is equivalent, as a monoidal category, to the category of representations of $\mathfrak{S}_{k}$.

The proof of Proposition 6.4 will be done by induction on $r$. The proof for the case $r = 1$ is based on a symmetry result obtained in [CEE], see also [EGL]. There is a symmetry of parameters for the simple quotients of spherical rational Cherednik algebras. Namely, for positive integers $n,N$ (not necessarily coprime) consider the Cherednik algebras $H^{c}_{N/n}(n)$ and $H^{c}_{n/N}(N)$, with maximal ideals $J_{\mathrm{max}}$ and $J'_{\mathrm{max}}$, respectively. Both parameters are spherical so $eJ_{\mathrm{max}}e, e'J'_{\mathrm{max}}e'$ are the maximal ideals of the spherical Cherednik algebras $eH^{c}_{N/n}(n)e,e'H^{c}_{n/N}(N)e'$, respectively. Here, $e \in \mathbb{C}S_{n}$ and $e' \in \mathbb{C}S_{N}$ denote the trivial idempotents in their respective group algebras. Then, by [CEE Prop. 9.5], [EGL Prop. 7.7], we have an isomorphism between the algebras $eH^{c}_{N/n}(n)e,e'H^{c}_{n/N}(N)e'$, mapping (the images of) the subalgebras $\mathbb{C}[\mathfrak{h}^{n}_{e}]^{S_{n}}, \mathbb{C}[\mathfrak{h}^{n}_{e}]^{S_{n}}$ to (the images of) the subalgebras $\mathbb{C}[\mathfrak{h}^{n}_{e}]^{S_{n}}, \mathbb{C}[\mathfrak{h}^{n}_{e}]^{S_{n}}$, respectively. Since HC $eHe$-bimodules with minimal support are precisely the ones whose annihilator is the maximal ideal in $eHe$, we have the following easy consequence of the above mentioned results.

Proposition 6.5. The isomorphism $eH^{c}_{N/n}(n)e,eJ_{\mathrm{max}}e \cong e'H^{c}_{n/N}(N)e'e'J'_{\mathrm{max}}e'$ induces a tensor equivalence between the categories of minimally supported HC $eHe$-$n(n)$-bimodules and minimally supported HC $eHe$-$r/N$-bimodules.

Now, the parameter $c = r/m > 0$, with gcd$(r;m) = 1$ and $mk = n$ for $k \in \mathbb{Z}_{>0}$, is spherical for the rational Cherednik algebra associated to $\mathfrak{S}_{n}$. Then, Proposition 6.5 has the following consequence.

Corollary 6.6. The categories of minimally supported HC $H_{r/m}(n)$-bimodules and minimally supported HC $H_{m/r}(rk)$-bimodules are equivalent as monoidal categories.

Now the case $r = 1$ of Proposition 6.4 is an easy consequence of Corollary 6.6 and [BEG Theorem 8.5], that asserts that the category of HC $H^{c}_{m}(k)$-bimodules is equivalent, as a monoidal category, to the category of representations of $\mathfrak{S}_{k}$. To complete the proof of Proposition 6.4 we use an inductive argument for which
we will need the theory of shift functors for rational Cherednik algebras of type A, see e.g. [GS, Section 3]. Namely, consider the $e \cdot e_{c+1}(n)$-bimodule $Q_{c+1}^e := e \cdot e_{c+1}(n)$. Here, $e_{c+1}$ denotes the sign idempotent, $e_{\text{sign}} = \frac{1}{m} \sum_{\sigma \in S_n} \text{sign}(\sigma)$. The bimodule $Q_{c+1}^e$ is HC, this follows from [GGS, Theorem 1.7]. The functor $F : e \cdot e_{c+1}(n) \rightarrow e \cdot e_{c+1}(n)$ given by $F(M) = Q_{c+1}^e \otimes e \cdot e_{c+1}(n) M$ is then an equivalence of categories, [BE, Corollary 4.3]. A quasi-inverse functor is given by tensoring with the $(e \cdot e_{c+1}(n))$-bimodule $P_{c+1}^e := \delta_{c+1} e_{\text{sign}} e_{c+1}(n)$, see Section 3 in [GS] (we remark that [GS] assumes that $c \not\in \frac{1}{2} + \mathbb{Z}$, an assumption that was later removed in [BE, Corollary 4.3]). The bimodule $P_{c+1}^e$ is also HC. It then follows that we have an equivalence of monoidal categories $F : HC(e \cdot e_{c+1}(n)) \rightarrow HC(e \cdot e_{c+1}(n))$, $F(V) = Q_{c+1}^e \otimes e \cdot e_{c+1}(n) V \otimes e \cdot e_{c+1}(n) P_{c+1}^e$. Clearly, this equivalence preserves the filtrations of the categories of HC bimodules by the support.

We now proceed to finish the proof of Proposition 6.4. So let $r, m, n, k$ be as in the statement of that proposition. We work over spherical subalgebras, and we make the following inductive assumption:

For every $0 < r' < r$ and every $m', k' \in \mathbb{Z}_{>0}$ with $\gcd(r', m') = 1$, the category $HC_{\mathcal{L}_k}(e \cdot H_{r'/m'}(m'k')e)$ is equivalent, as a monoidal category, to the category of representations of $\mathcal{S}_k$.

Clearly, Proposition 6.5 together with [BE, Theorem 8.5], give the base of induction. Now, using Proposition 6.5 again, we have that the categories $HC_{\mathcal{L}_k}(e \cdot H_{r/m}(n)e)$ and $HC_{\mathcal{L}_k}(e' \cdot H_{r/m}(e)''(k'))$ are equivalent as monoidal categories. Using shift functors, we get a tensor equivalence between $HC_{\mathcal{L}_k}(e \cdot H_{r/m}(n)e)$ and $HC_{\mathcal{L}_k}(e' \cdot H_{r/m}(e)''(k'))$, where $0 < r' < r$. By our inductive assumption, this is tensor equivalent to $\mathcal{S}_k$-rep.

Proposition 6.4 now follows by sphericity, since we are assuming our parameter $c$ is positive.

Let us give a description of the irreducible objects in $HC_{\mathcal{L}_k}(H_{r/m}(n))$. First of all, the irreducible modules in $\mathcal{O}_{r/m}$ have the form $L(m\lambda)$, where $\lambda$ is a partition of $k$. The irreducible module that gets sent to the trivial $\mathcal{S}_k$-representation under KZ is $L(m \cdot \text{triv}_k) = L(\text{triv})$, where $\text{triv}_k$ stands for the trivial partition of $k$ and $\text{triv} = m \cdot \text{triv}_k$ is the trivial partition of $n$. The localization $L_{r/m}^k$ of the bimodule $Hom_{\mathcal{L}_k}(L(\text{triv}), L(m\lambda))$ is irreducible, so each one of $Hom_{\mathcal{L}_k}(L(\text{triv}), L(m\lambda))$ is either irreducible or 0. But a HC bimodule $B \in HC_{\mathcal{L}_k}(H_{r/m}(n))$ is a Noetherian bimodule over the simple algebra $H_{r/m}$, so it is a progenerator in the category of left and right modules over this algebra, cf. [Be, Theorem 10]. In particular, $B \otimes_{H_{r/m}} L(\text{triv}) \neq 0$, so any irreducible HC bimodule with minimal support embeds in a bimodule of the form $Hom_{\mathcal{L}_k}(L(\text{triv}), L(m\lambda))$. By counting, it follows that $\{Hom_{\mathcal{L}_k}(L(\text{triv}), L(m\lambda)) : \lambda \text{ is a partition of } q \}$ is a complete list of irreducible HC $H_{r/m}(n)$-bimodules with minimal support. An explicit tensor equivalence is given as follows, $B = Hom_{\mathcal{L}_k}(L(\text{triv}), L(m\lambda)) \mapsto KZ_B^k(B \otimes_{H_{r/m}} L(\text{triv}))$. That this functor intertwines tensor products follows by an analog of Lemma 4.8 using the functor $KZ_B^k$ instead of KZ.

Remark 6.7. We remark that, while $\{Hom_{\mathcal{L}_k}(L(\text{triv}), L(m\lambda)) : \lambda \vdash k\}$ forms a complete and irredundant list of irreducible HC bimodules with minimal support, we have that $Hom_{\mathcal{L}_k}(L(m\mu), L(m\lambda)) = 0$ where $\lambda, \mu$ are any partitions of $k$. This follows from, for example, Theorem 8.16 in [BE], which gives a description of $Hom_{\mathcal{L}_k}(L(m\mu), L(m\lambda))$ as a direct sum of bimodules of the form $Hom_{\mathcal{L}_k}(L(\text{triv}), L(m\xi))$.

To finish this subsection, let us explain what happens when we have $c = r/m < 0$, with $\gcd(r, m) = 1$ and $m$ divides $n$, say $km = n$. In this case, the category $HC_{\mathcal{L}_k}(n)$ is also equivalent to the category of representations of $\mathcal{S}_k$. This follows because there is an equivalence $HC_{\mathcal{L}_k}(H_{r/m}(n)) \cong HC_{\mathcal{S}_k}(H_{r/m}(n))$ induced by an isomorphism $H_{r/m}(n) \rightarrow H_{-c}(n)$, mapping $h^n \ni x \mapsto x$, $h \ni y \mapsto y$, $\mathcal{S}_n \ni \sigma \mapsto \text{sign}(\sigma)\sigma$.

6.4. Irreducible HC bimodules. We use the results of the previous subsection and Section 3 to give a classification of all irreducible HC $H_{r/m}(n)$-bimodules where, as above, we assume that $c$ has the form $c = r/m > 0$, with $1 < m \leq n$ and $\gcd(r, m) = 1$. The following is the main result of this subsection.

Theorem 6.8. Let $c = r/m > 0$, with $1 < m \leq n$, $\gcd(r, m) = 1$, and let $i = 1, \ldots, \lfloor n/m \rfloor$. Then, the category $HC_{\mathcal{L}_i}(H_{r/m}(n))$ is equivalent to the category of representations of $\mathcal{S}_i$.

Before proceeding to the proof of Theorem 6.8 we describe the objects in the category $HC_{\mathcal{S}_0}^{\mathcal{S}_0}(H_{r/m})$. Recall that this category is equivalent to the category of representations of $\mathcal{Z} = \mathcal{S}_0 \times \mathcal{S}_i$, this follows because the algebra $H_{r/m}$ has a unique irreducible finite dimensional bimodule (that does not admit non-trivial self-extensions). This bimodule is $B := Hom_{\mathcal{S}_0}(L(\text{triv}_W), L(\text{triv}_W))$, this is a consequence of the fact that
$L(\text{triv}_W)$ is the unique irreducible finite dimensional module over the algebra $H_c$. Moreover, since $c = r/m$ and $\bar{W} = \mathcal{S}_i^{\times 2}$, we have that $H_c = H_r(m)^{\otimes 2}$, and $\mathcal{B} = \mathcal{B}^{\otimes 2}$, where $\mathcal{B}$ is the unique irreducible finite dimensional bimodule over $H_c(m)$, so $\mathcal{B}$ admits a $\Xi$-equivariant structure, where $\mathcal{S}_i$ permutes the tensor factors and $\mathcal{S}_{n-m}$ acts trivially. Under the equivalence $\text{HC}_0^\Xi(H_c) \to (\mathcal{S}_i \times \mathcal{S}_{n-m})\text{-rep}$, $\mathcal{B}$ corresponds to the trivial representation. So we have the following result.

**Lemma 6.9.** The irreducible objects in $\text{HC}_0^\Xi(H_c)$ have the form $\mathcal{B} \otimes \xi$, where $\xi$ runs over the set of irreducible representations of $\mathcal{S}_i \times \mathcal{S}_{n-m}$, which acts diagonally. The irreducibles where $\mathcal{S}_{n-m}$ acts trivially correspond precisely to those representations $\mathcal{B} \otimes \xi$ where $\xi$ factors through $\mathcal{S}_i$.

**Proof of Theorem 6.8.** We need to check that, if $\mathcal{B}$ is an irreducible representation of $\mathcal{S}_i$, then the equivariant bimodule $\mathcal{B} \otimes \xi$ belongs to the image of $\otimes_{\mathcal{S}_i}$. By Corollary 3.11 for every parabolic subgroup $W$ containing $\mathcal{W}$ in corank 1 we need to produce a $\mathcal{H} \mathcal{C}_c(W')$-bimodule $B'$ with $B'_1 = \mathcal{B} \otimes \xi$, with the restricted $\mathcal{H}_c(W')$-equivariant structure. The subgroups $W'$ have three different types. Either $W' \cong \mathcal{S}_m^{(i-2)} \times \mathcal{S}_2$, where $N_{W'}(W)/W \cong \mathcal{S}_2$, acting on $H_c = H_r/m(m)^{\otimes 2}$ by permuting two of the tensor factors. Thanks to the results of Subsection 6.3 the functor $\otimes_{\mathcal{S}_i}$, $\mathcal{H} \mathcal{C}_c(W') \to \mathcal{H} \mathcal{C}_0^\Xi(H_c)$ is essentially surjective, were $\mathcal{H} \mathcal{C}_c(W')$ denotes the category of minimally supported $H_c(W')$-bimodules. So we can certainly find a bimodule $B'$ with $B'_1 = B \otimes \xi$.

**Case 1.** $W' \cong \mathcal{S}_m^{(i-2)} \times \mathcal{S}_{2m}$, so that $N_{W'}(W)/W \cong \mathcal{S}_2$, acting on $H_c = H_r/m(m)^{\otimes 2}$ by permuting two of the tensor factors. Thanks to our assumptions on $\xi$, $\mathcal{S}_2$ acts trivially on $\mathcal{B}$. It also acts trivially on $\mathcal{B}$. So we need to check that $\mathcal{B}$, with trivial action of $\mathcal{S}_2$, belongs to the image of $\otimes_{\mathcal{S}_i}$. Upon the identification $\mathcal{H} \mathcal{C}_0^\Xi(H_c) \to \mathcal{S}_2\text{-rep}$, $\mathcal{B}$ corresponds to the trivial representation. The image of the restriction functor is closed under tensor products and sub-bimodules. The trivial representation of $\mathcal{S}_2$ is contained in $S^\otimes 2$ for any representation $S$ of $\mathcal{S}_2$, the result follows.

**Case 2.** $W' \cong \mathcal{S}_m^{(i-1)} \times \mathcal{S}_{m+1}$, so that $N_{W'}(W)/W \cong \{1\}$. Thus, what we have to check here is that $\mathcal{B}$ belongs to the image of the functor $\otimes_{\mathcal{S}_i}$. But this follows because the image of $\otimes_{\mathcal{S}_i}$ is closed under sub-bimodules.

**Case 3.** $W' \cong \mathcal{S}_m \times \mathcal{S}_2$, so that $N_{W'}(W)/W \cong \mathcal{S}_2$, acting trivially on $H_c$. Thanks to our assumptions on $\xi$, $\mathcal{S}_2$ acts trivially on $\mathcal{B}$. It also acts trivially on $\mathcal{B}$. So we need to check that $\mathcal{B}$, with trivial action of $\mathcal{S}_2$, belongs to the image of $\otimes_{\mathcal{S}_i}$. Upon the identification $\mathcal{H} \mathcal{C}_0^\Xi(H_c) \to \mathcal{S}_2\text{-rep}$, $\mathcal{B}$ corresponds to the trivial representation. The image of the restriction functor is closed under tensor products and sub-bimodules. Since the trivial representation of $\mathcal{S}_2$ is contained in $S^\otimes 2$ for any representation $S$ of $\mathcal{S}_2$, the result follows.

**Proposition 6.10.** The following is true:

(i) $\text{Ext}(H_c, M) = 0$.
(ii) $\text{Ext}(M, H_c) = 0$.
(iii) $\text{dim}(\text{Ext}(M, J)) = 1$.
(iv) $\text{Ext}(H_c, J) = 0$.
(v) $\text{Ext}(\mathcal{J}, H_c) = 0$.
(vi) $\text{Ext}(M, D) = 0$.
(vii) $\text{Ext}(D, J) = 0$.
(viii) $\text{Ext}(D, M) = 0$.
(ix) $\text{Ext}(\mathcal{J}, D) = 0$. 

6.5. Case $c = r/n$. In this subsection, we completely characterize the category of HC bimodules over the algebra $H_{r/n}(n)$, where $\gcd(r; n) = 1$. By Theorem 5.3 this algebra has a unique irreducible HC bimodule with full support, namely, the unique nonzero proper ideal $\mathcal{J} \subseteq H_c$. By Subsection 6.2 this bimodule does not have self-extensions. On the other hand, this algebra has a unique irreducible finite dimensional bimodule, namely $M := H_{c/\mathcal{J}}$, that does not admit self-extensions. The bimodules $M, \mathcal{J}$ form a complete list of irreducible HC bimodules. We now investigate extensions between them. For the rest of this section we denote simply by ‘Ext’ the extension group $\text{Ext}_{H_c}\text{-bimod}$. It is clear that $\text{Ext}(M, \mathcal{J}) \neq 0$, as $H_c$ is a non-split extension of $\mathcal{J}$ by $M$. On the other hand, [BL] Subsection 7.6 constructs a non-split extension $D$ $M$ by $\mathcal{J}$. Our goal now is to show that $M, \mathcal{J}, H_c$ and $D$ form a complete list of indecomposable HC $H_c$-bimodules. This is a consequence of the following result.

**Proposition 6.10.** The following is true:

(i) $\text{Ext}(H_c, M) = 0$.
(ii) $\text{Ext}(M, H_c) = 0$.
(iii) $\text{dim}(\text{Ext}(M, \mathcal{J})) = 1$.
(iv) $\text{Ext}(H_c, \mathcal{J}) = 0$.
(v) $\text{Ext}(\mathcal{J}, H_c) = 0$.
(vi) $\text{Ext}(M, \mathcal{J}) = 0$.
(vii) $\text{Ext}(\mathcal{J}, D) = 0$.
(viii) $\text{Ext}(D, M) = 0$.
(ix) $\text{Ext}(\mathcal{J}, D) = 0$.
(x) \( \dim(\text{Ext}(\mathcal{J}, M)) = 1 \).

Proof. We show that (i) holds more generally. Namely, we have the following result.

**Lemma 6.11.** Let \( H_c \) be any rational Cherednik algebra of type A (we do not put restrictions on the parameter \( c \)), and let \( M \) be an irreducible Harish-Chandra \( H_c \)-bimodule with minimal support. Then, \( \text{Ext}(H_c, M) = 0 \).

Proof. We know that \( \text{Ext}^\bullet(H_c, M) = \text{HH}^\bullet(H_c, M) \), where \( \text{HH}^\bullet \) denotes Hochschild cohomology, so we need to compute \( \text{HH}^1(H_c, M) \). It is well known that this is the space of outer derivations (i.e., the space of derivations modulo the space of inner derivations). Now, let \( \delta : H_c \rightarrow M \) be a derivation. Since \( \mathcal{J}^2 = \mathcal{J} \), Subsection 6.1, the Leibniz rule implies that \( \delta(\mathcal{J}) = 0 \), so \( \delta \) factors through the quotient algebra \( H_c/\mathcal{J} \). This implies that \( \text{HH}^1(H_c, M) = \text{HH}^1(H_c/\mathcal{J}, M) \) (note that \( M \) is an \( H_c/\mathcal{J} \)-bimodule since \( \text{RAnn}(M) = \text{LAnn}(M) = \mathcal{J} \), so this last Hochschild cohomology does make sense). Now, both \( H_c/\mathcal{J} \) and \( M \) are irreducible HC bimodules with minimal support. Recall, Subsection 6.2, that the category of HC bimodules with minimal support is semisimple. Then, \( \text{Ext}(H_c/\mathcal{J}, M) = 0 \), which implies that \( \text{HH}^1(H_c/\mathcal{J}, M) = \text{Ext}_{H_c/\mathcal{J}-\text{bimod}}(H_c/\mathcal{J}, M) = 0 \).

Then (i) is a special case of Lemma 6.11. Note that (ii) and (v) are consequences of Proposition 2.10. Now (iii) is a consequence of (ii): we have a long exact sequence

\[
0 \rightarrow \text{Hom}(M, \mathcal{J}) \rightarrow \text{Hom}(M, H_c) \rightarrow \text{Hom}(M, M) \rightarrow \text{Ext}(M, \mathcal{J}) \rightarrow \text{Ext}(M, H_c) \rightarrow \cdots
\]

Now, both \( \text{Hom}(M, H_c) \) and \( \text{Ext}(M, H_c) \) are 0, so \( \text{Hom}(M, M) \rightarrow \text{Ext}(M, \mathcal{J}) \) must be an isomorphism and the claim follows. Again using long exact sequences, we can see that \( \text{dim}(\text{Ext}(H_c, M)) = -\text{dim}(\text{Hom}(\mathcal{J}, M)) + \text{dim}(\text{Ext}(M, \mathcal{J})) = 0 \), so (iv) is proved. Statements (vi), (ix) are consequences of the following.

**Proposition 6.12.** Assume \( c = r/n \), with \( \gcd(r; n) = 1 \). The bimodule \( D \) is injective in the category of HC \( H_c(n) \)-bimodules.

Proof. We remark that we have functors \( F : \text{HC}(c, c) \rightarrow \mathcal{O}_c, B \mapsto B \otimes_{H_c} \Delta_c(\text{triv}), G : \mathcal{O}_c \rightarrow \text{HC}(c, c), M \mapsto \text{Hom}_{\mathfrak{f}1n}(\Delta_c(\text{triv}), M) \). Since \( c > 0 \), \( \Delta_c(\text{triv}) \) is projective in \( \mathcal{O}_c \), so thanks to [LA] Lemma 3.9 the functor \( F \) is exact. Note also that \( F \) is left adjoint to \( G \), so \( G \) maps injective objects to injective objects. Thus, the lemma will follow if we find an injective \( N \in \mathcal{O}_c \) with \( D = \text{Hom}_{\mathfrak{f}1n}(\Delta_c(\text{triv}), N) \). Thanks to [BEC2 Corollary 1.3], \( \Delta_c(\text{triv}) \) has a unique proper nonzero submodule, say \( I \). We claim that \( \text{Hom}_{\mathfrak{f}1n}(\Delta_c(\text{triv}), I) = \mathcal{J} \), this follows because \( I = \mathcal{J} \Delta_c(\text{triv}) \) and Corollary 4.6. We also have that \( \text{Hom}_{\mathfrak{f}1n}(\Delta_c(\text{triv}), L_c(\text{triv})) = \text{Hom}_{\mathcal{O}_c}(L_c(\text{triv}), L_c(\text{triv})) = H_c/\mathcal{J} = M \).

The costandard module \( \nabla_c(\text{triv}) \) is injective in category \( \mathcal{O}_c \), it has a unique proper submodule isomorphic to \( L_c(\text{triv}) \) and \( \nabla_c(\text{triv})/L_c(\text{triv}) \cong I \), all of these properties follow from the construction of \( \nabla_c(\text{triv}) \), see e.g. [GGOR Subsection 2.3]. So \( G(\nabla_c(\text{triv})) \) is injective and contains \( \text{Hom}_{\mathfrak{f}1n}(\Delta_c(\text{triv}), L_c(\text{triv})) = M \). It follows that we have an injection \( D \rightarrow G(\nabla_c(\text{triv})) \). Note, however, that we have an exact sequence \( 0 \rightarrow G(L_c(\text{triv})) \rightarrow G(\nabla_c(\text{triv})) \rightarrow G(I) \). Thanks to the previous paragraph, we conclude that the composition length of \( G(\nabla_c(\text{triv})) \) is \( \leq 2 \). So \( D \cong \text{Hom}_{\mathfrak{f}1n}(\Delta_c(\text{triv}), \nabla_c(\text{triv})) \) and is therefore injective.

**Remark 6.13.** It is worth noticing that the category \( \text{HC}(H_c(W)) \) has enough injectives for any complex reflection group \( W \) and parameter \( c \). Indeed, let \( P_c \) be a progenerator of the category \( \mathcal{O}_c \). Thanks to Lemma 3.9 in [LA] the functor \( F : \text{HC}(H_c(W)) \rightarrow \mathcal{O}_c, B \mapsto B \otimes_{H_c} P_c \) is exact. Moreover, \( F \) admits a right adjoint \( G : \mathcal{O}_c \rightarrow \text{HC}(H_c), M \mapsto \text{Hom}_{\mathfrak{f}1n}(P_c, M) \). So \( G \) has to map injectives to injectives. Thanks to [LA] Lemma 3.10, every irreducible \( H_c \)-bimodule is contained in one of the form \( G(M) \) for some \( M \in \mathcal{O}_c \). This implies that there are enough injectives in \( \text{HC}(H_c) \). When \( W = S_n \) and \( c > 0 \), we can replace \( P_c \) by \( \Delta_c(\text{triv}) \). This follows because \( \Delta_c(\text{triv}) \) is projective and the results in Subsection 6.2 imply that every irreducible HC bimodule is contained in one of the form \( \text{Hom}_{\mathfrak{f}1n}(\Delta_c(\text{triv}), M) \).

Now we show that \( \text{Ext}(D, \mathcal{J}) = 0 \). Assume we have a short exact sequence

\[
0 \rightarrow \mathcal{J} \rightarrow X \xrightarrow{\pi} D \rightarrow 0
\]
Consider the induced exact sequence \( 0 \to J \to \pi^{-1}(M) \to M \to 0 \). So either \( \pi^{-1}(M) = H_c \) or \( \pi^{-1}(M) = J \oplus M \). If \( \pi^{-1}(M) = H_c \), then the exact sequence \( 0 \to \pi^{-1}(M) \to X \to J \to 0 \) gives \( X = H_c \oplus J \), cf. (iv), which contradicts the existence of the exact sequence (6). Then, we must have \( \pi^{-1}(M) = J \oplus M \). Using again the exact sequence \( 0 \to \pi^{-1}(M) \to X \to J \to 0 \), we get that \( X = J \oplus V \), where \( V \) is an extension of \( M \) by \( J \). Then (6) forces \( V = D \) and the sequence splits.

The proof of (viii) is similar: say that we have a short exact sequence

\[
\begin{align*}
0 & \to M \to X \to D \\
\end{align*}
\]

So we see that \( \text{Soc}(X) = M \oplus M \) and we have an exact sequence

\[
0 \to M \oplus M \to X \to J \to 0
\]

An extension of \( M \oplus M \) by \( J \) must be of the form \( B \oplus M \), where \( B \) is an extension of \( M \) by \( J \). Using the short exact sequence (7) we see that \( X \cong D \oplus M \). Finally, (x) is an easy consequence of the previous statements. □

Note that the previous proposition implies that both \( H_c \) and \( D \) are injective-projective in the category \( 
\text{HC}(c, c) \). The injective hull of \( J \) coincides with the projective cover of \( M \), which is \( H_c \), while \( D \) is both the injective hull of \( M \) and the projective cover of \( J \). It follows, in particular, that the homological dimension of \( 
\text{HC}(c, c) \) is infinite.

**Remark 6.14.** The results of this subsection are also valid for parameters of the form \( c = r/m \), with \( \gcd(r; m) = 1 \) and \( |n/2| < m \leq n \). Indeed, here we also have two irreducible \( 
\text{HC} \) bimodules \( M \) and \( J \), where \( J \) has full support and \( M \) has minimal support. Proposition 6.10 is valid with the same proof, so in this case the category \( 
\text{HC}(c, c) \) is equivalent to the category of representations of the quiver

\[
\begin{array}{c}
\bullet \\
\alpha \searrow & \nearrow \beta \end{array}
\]

with relations \( \alpha\beta = \beta\alpha = 0 \).

### 6.6. Two-parametric case.

We study the category \( 
\text{HC}(c, c') \) when the parameters \( c, c' \) are distinct. First, we remark that if \( c \) is a regular parameter (this means that \( c \) is not of the form \( r/m \) with \( 0 < m \leq n \)) then \( 
\text{HC}(c, c') = 0 \) unless \( c' = \pm c + m \) with \( m \in \mathbb{Z} \) and, in this case, \( 
\text{HC}(c, c') \) has been completely described, Theorem 1.1. So we may assume that both parameters \( c, c' \) are singular. Since for irreducible modules \( M \in \mathcal{O}_{c'}, N \in \mathcal{O}_c \), \( \text{Hom}_{\mathcal{F}}^\text{fin}(M, N) \neq 0 \) only when \( \text{supp}(M) = \text{supp}(N) \), the description of supports of irreducible modules given in Subsection 6.1 implies that a necessary condition for \( 
\text{HC}(c, c') \) to be nonzero is that \( c \) and \( c' \) have the same denominator when expressed as irreducible fractions. Then, throughout this subsection we assume that \( c = r/m, c' = r'/m, \gcd(r; m) = \gcd(r'; m) = 1 \), \( 1 < m \leq n \).

Recall that, for \( i = 1, \ldots, [n/m] \) we have the functor \( KZ_{c'}^i : \mathcal{O}_{c'}^i \to (\mathbb{C}\mathcal{G}_i \otimes \mathcal{H}_q((\mathcal{G}_{n-mi})^\text{mod}, \text{mod}) \text{ mod} \), where \( q' = \exp(2\pi \sqrt{-1}c') \). Let \( N \in \mathcal{O}_{c'}^i \) be the irreducible module with \( KZ_{c'}^i(N) = \text{triv} \). Then, similarly to Section 4, we have that every irreducible \( \text{HC}_{c}(n) - \text{HC}_{c'}(n) \)-bimodule supported on the closure of the symplectic leaf \( \mathcal{L}_i \) is contained in a bimodule of the form \( \text{Hom}_{\mathcal{F}}^\text{fin}(N, M) \) for an irreducible module \( M \in \mathcal{O}_c^i \) and, moreover, that whenever \( \text{Hom}_{\mathcal{F}}^\text{fin}(N, M) \) is nonzero then, for every module \( L \in (\mathbb{C}\mathcal{G}_i \otimes \mathcal{H}_q(n)) \text{ mod} \) the \( B_{n-mi} \)-module \( KZ_i'(M) \otimes C L \) factors through the algebra \( \mathbb{C}\mathcal{G}_i \otimes \mathcal{H}_q(n-mi) \). The following result is then completely analogous to Proposition 6.2.

**Proposition 6.15.** Let \( i \in \{1, \ldots, [n/m]\} \) and assume that \( n - mi \neq 0 \). Then, \( \text{HC}_{\mathcal{L}_i}(c, c') = 0 \) unless \( c - c' \in \mathbb{Z} \) or \( c + c' \in \mathbb{Z} \).

Now assume that \( c' = c + k \), with \( k > 0 \) and \( k \in \mathbb{Z}_{>0} \). Then, using shift functors we have an equivalence of categories \( \text{HC}(c, c) \cong \text{HC}(c, c') \cong \text{HC}(c', c') \) preserving the filtration by supports so that, in particular, they descend to equivalences \( \text{HC}_{\mathcal{L}_i}(c, c) \cong \text{HC}_{\mathcal{L}_i}(c, c') \cong \text{HC}_{\mathcal{L}_i}(c', c') \). Since we have an isomorphism \( H_c(n) \to H_{c'}(n) \) fixing the subalgebras \( \mathbb{C}[h]^\text{fin}, \mathbb{C}[h]^\text{fin} \), we also have an equivalence \( \text{HC}(c, -c - k) \cong \text{HC}(c, c) \) preserving the filtration by supports. Similar results hold if \( c < 0 \) and \( k \in \mathbb{Z}_{<0} \).
Assume now that $c = c' + 1$, with $-1 < c' < 0$. In this case, the shift functor is not an equivalence. However, it does induce a derived equivalence $\kappa_{c'} \circ \kappa_c : D^b(H_c, \mathfrak{h}_c) \to D^b(H_{c'}, \mathfrak{h}_{c'})$, see e.g. [GL] Section 5. It follows that, if we denote by $D^b_{HC}(c, c')$ the subcategory of $D^b(H_c, \mathfrak{h}_c)$ consisting of complexes with HC homology, then we have a derived equivalence $\kappa_{c'} \circ \kappa_c : D^b_{HC}(c, c') \to D^b_{HC}(c, c')$. Thus, the categories $HC(c, c)$ and $HC(c, c')$ have the same number of irreducibles. We remark here that Proposition 6.3 is valid in the two-parametric setting, with the same proof. Hence, $HC_{L_t}(c, c') \cong \mathfrak{S}_t$-rep for $i = 1, \ldots, [n/m]$. The same holds for the category $HC_{L_t}(c', c)$.

References


Department of Mathematics, Northeastern University. Boston, MA 02115. USA.
E-mail address: simentalrodriguez.j@husky.neu.edu