Alpha-Type and Exhaustive Weakly Wandering Sequences for Ergodic Infinite Measure Preserving Transformations

2007-2008
Throughout this talk

\((T, X, \mathcal{B}, \mu)\)

- \((X, \mathcal{B}, \mu)\): \(\sigma\)-finite, nonatomic, Lebesgue Space
- \(\mu(X) = \infty\)
- \(T\): Ergodic, Invertible, Measure-Preserving Transformation
Definition

- \( \{n_i : i = 0, 1, 2, \ldots \} \) a sequence of integers (\( n_i \neq n_j \) for \( i \neq j \))
- \( W \) a measurable set in \( X \)

are **Exhaustive Weakly Wandering** for \( T \) if

\[
X = \bigcup_{i=0}^{\infty} T^{n_i} W \quad (\text{Exh.})
\]

\[
T^{n_i} W \cap T^{n_j} W = \emptyset, \quad i \neq j \quad (W.W.)
\]

\( \{n_i\} \) is the **Ex.W.W. Sequence**.

\( W \) is the **Ex.W.W. Set**.
Theorem (Jones - Krengel)

Every ergodic, infinite measure preserving transformation has Exhaustive Weakly Wandering sets and sequences.

However, for many (most) transformations no Exhaustive Weakly Wandering set or sequence is known.
**Definition**

*T* is of *α*-type, for $0 \leq \alpha \leq 1$, if

$$\limsup_{n \to \infty} \mu(T^n A \cap A) = \alpha \cdot \mu(A)$$

for all $A$ satisfying $\mu(A) < \infty$. 
To date, the only known Alpha-types for which specific Exhaustive Weakly Wandering sequences and sets are explicitly known are for

\[ \alpha = \frac{n - 1}{n} \]

To date, no Exhaustive Weakly Wandering sequence is known to work for two different Alpha-Types.
Theorem

There exists a sequence $\mathbb{B} = \{b_0, b_1, \cdots\}$. $b_i \neq b_j$ for $i \neq j$ such that there exists a family of maps $T_\alpha$, $0 \leq \alpha \leq 1$ so that for all $\alpha$

- $T_\alpha$ is ergodic infinite measure preserving
- $\mathbb{B}$ is an Exhaustive Weakly Wandering sequence for all $T_\alpha$
- All the Exh.W.W. sets $W_\alpha$ have measure one
- $T_\alpha$ is of $\alpha$-type
The sequence $\mathbb{B}$ is a Direct Sum (Finite Sum of Finite Sets)

\[
\mathbb{B} = \bigoplus_{n=1}^{\infty} \mathbb{B}_n
\]

$\mathbb{B}_1 = \{0, 2\}$
$\mathbb{B}_2 = \{0, 16, 32, 48, 64, 80, 96, 112\}$
$\mathbb{B}_3 = \{0, 1024, 2048, 3072, 4096, 5120 \cdots 130048\}$
$
\vdots$

$\mathbb{B}_k = \{0, b_k, 2 \cdot b_k, 3 \cdot b_k, \cdots, (2^{2^k-1} - 1) \cdots b_k\}$

where $b_k = 2^k \cdot \prod_{j=0}^{k-1} 2^{2^j+j-1}$
Ohio-State - An Example of Hajian-Kakutani

The fundamental ideas of the proofs can be illustrated with a simple example.

In fact it is the first known example for the two concepts of Alpha-type and Exhaustive Weakly Wandering.

Specifically,

- Alpha-type for $\alpha = \frac{1}{2}$
- The Ex.W.W. set is displayable and has measure 1
- The Ex.W.W. sequence is definable and is

$$\mathbb{B} = IP\{2^{2n+1}\} = \{0, 2, 8, 10, \cdots\}$$

0 corresponds to the empty sum
The example will be presented Three ways.

1. a Cutting, Stacking, Spacer construction
2. a Skyscraper construction
3. an Adding Machine construction

1. The Cutting, Stacking, Spacer construction will allow the Alpha-type to be controlled.
2. The Skyscraper construction will allow the Exhaustive Weakly Wandering set to be displayed.
3. The Adding Machine construction will allow control of the Exhaustive Weakly Wandering sequence.
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Construction - Cut, Add Spacers, Stack

Stage 1: Start with the Interval $[0, 1)$
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Cut in half
Construction - Cut, Add Spacers, Stack

Stage 1: Start with the Interval \([0, 1)\)

Cut in half

Add 2 spacers - on Right Column
Construction - Cut, Add Spacers, Stack

Stage 1: Start with the Interval \([0, 1)\)

Cut in half

Add 2 spacers - on Right Column

Stack Right-column onto Left
Construction - Cut, Add Spacers, Stack

Stage 1: Start with the Interval $[0, 1)$

Cut in half

Add 2 spacers - on Right Column

Stack Right-column onto Left

The Transformation $T$ goes up

---------

-------- $T$

---------
Construction - Cut, Add Spacers, Stack

Stage 2: Start with previous column
Construction - Cut, Add Spacers, Stack

Stage 2: Start with previous column

Cut in half
Construction - Cut, Add Spacers, Stack

Stage 2: Start with previous column

Cut in half

Add $2^3$ spacers - on Right Column
Construction - Cut, Add Spacers, Stack

Stage 2: Start with previous column

Cut in half

Add $2^3$ spacers - on Right Column

Stack Right-column onto Left
Construction - Cut, Add Spacers, Stack

Stage 2: Start with previous column

Cut in half

Add $2^3$ spacers - on Right Column

Stack Right-column onto Left

The Transformation $T$ goes up
$n^{th}$ Stage of Construction

1. Cut previous column in half.
2. Add $2^n$ spacers on right subcolumn.
3. Stack right column on top of left column.
4. $T$ defined going up the column.
$W = [0, 1)$ is the Exhaustive Weakly Wandering Set

At Stage 1 we can "see" $W$ and $T^2 W$ "filling" the space
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At Stage 1 we can "see" \( W \) and \( T^2W \) "filling" the space

At Stage 2, \( W, T^2W \) and \( T^8W \) fill the space
$W = [0, 1)$ is the Exhaustive Weakly Wandering Set

At Stage 1 we can "see" $W$ and $T^2 W$ "filling" the space

At Stage 2, $W$, $T^2 W$ and $T^8 W$ $T^{10} W$ "fill"
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Exhaustive Weakly Wandering Sequence for Ohio-State

So far the Ex.W.W. sequence for the Ohio-State is

\( \{0, 2, 8, 10\} \)

This can be derived as

\( \{0, 2\} \oplus \{0, 8\} \)
Exhaustive Weakly Wandering Sequence for Ohio-State

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The Exhaustive Weakly Wandering Sequence for the Ohio State Example is a Direct Sum.

\[ \bigoplus_{n=1}^{\infty} \{0, 2^{2n-1}\} = \bigoplus_{n=0}^{\infty} 2^{2n} \{0, 2\} \]

\[ 2^{2n-1} \] is the number of spacers being added at the \( n^{th} \) stage.
Alpha = $\frac{1}{2}$ for the Ohio State example

We illustrate on $W = [0, 1)$

$$m(W \cap T^{2n} W) = \frac{1}{2}$$

Otherwise

$$m((W \cap T^k W) < \frac{1}{2}$$
Alpha = \frac{1}{2} for the Ohio State example

We illustrate on \( W = [0, 1) \)

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Stage 1

\[ \overline{\bar{W}} \]
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Stage 1

$W \cap T^1 W$

Stage 2
Alpha = \frac{1}{2} for the Ohio State example

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\]

Stage 1

\[
W \cap T^1 W
\]

Stage 2

\[
W \cap T^4 W
\]
For the general construction we only need to define a sequence of Cuts and a sequence of Spacers to place atop the sub-columns before stacking (right on top of left).

- **The Number of Cuts**
  \[ m_n \geq 2 \]

- **The Number of Spacers**
  \[ S_n = \{ s_{1}^{(n)}, \ldots, s_{c_n}^{(n)} \} \]
Theorems - Sufficiency for Alpha-Types

**Theorem**

*If in the general construction (described above)*

- $m_k > m_{k-1}$, the cuts are increasing
- $s_{n+1}^{(k)} \geq \sum_{i=1}^{n} s_i^{(k)} + n \cdot h_{k-1}, \ n \geq 1$

*Then $T$ is of 0-type.*

The first spacer $s_1^{(k)} \geq 0$ is arbitrary - in practice it will be set to 0.
If in the general construction

- $m_k > m_{k-1}$
- $n_k < m_k$ such that $\lim_{k \to \infty} \frac{n_k}{m_k} = \alpha$
- $s_i^{(k)} = 0$ for $1 \leq i \leq n_k$
- $s_{j+1}^{(k)} \geq \sum_{i=1}^{j} s_i^{(k)} + j \cdot h_{k-1}, j \geq n_k$

Then $T$ is of $\alpha$-type

Note, the first $n_k$ need only be equal. We set them to 0 for simplicity.

The first theorem follows from the second - but all the concepts can be seen in the proof of the first.
Adding Machine Construction

If we don’t stack the columns, we see a Skyscraper.
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\[ W = [0, 1) \text{ is the Base.} \]

\[ \text{The Induced Transformation } T_W \text{ on } W \]

“is” the Dyadic Adding Machine
Ohio State - Reconstructed

\[ \Omega = \prod_{i=0}^{\infty} \{0, 1, 2, 3\} \]

\[ \tau = " + 1" \text{ (carry to the right)}. \]

\[ W = \{ \omega \in \Omega : \omega_i \in \{0, 1\} \} \subset \Omega \]

\[ X = \{ \omega : \exists N, \ \omega_i \in \{0, 1\} \ \forall i \geq N \} \]

\[ \omega = (000 \cdots) \in W \]

\[ \tau \omega = (100 \cdots) \in W \]

\[ \tau^2 \omega = (200 \cdots) \]

\[ \tau^3 \omega = (300 \cdots) \]

\[ \tau^4 \omega = (010 \cdots) \in W \]

\[ \tau^5 \omega = (110 \cdots) \in W \]

\[ \tau^6 \omega = (210 \cdots) \]

\[ \vdots \]
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