Categorification of quantum groups
and quantum knot invariants

Ben Webster

MIT/Oregon

March 17, 2010
The big picture

quantum groups $U_q(g)$

\[
\downarrow
\]

ribbon category of $U_q(g)$-reps

\[
\downarrow
\]

quantum knot polynomials
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Categorification of quantum knot invariants

March 17, 2010 2 / 29

The big picture

quantum groups $U_q(g)$ ←→
ribbon category of $U_q(g)$-reps ←→
quantum knot polynomials ←→ quantum knot homologies

HAVE

Khovanov-Lauda/Rouquier 2-categories $\mathcal{U}$

WANT

???
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categorifications of tensor products of simples

quantum knot homologies

 HAVE

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Morally, these knot invariants arise from Chern-Simons theory. They “are” the expectation value of the trace on a chosen representation of the holonomy around the knot for a certain probability distribution on the space of \( g \)-connections on \( S^3 \).

But I’d like to have a definition that didn’t require “are” to be in quotes.

What can be done is to make Chern-Simons theory an extended TQFT (attach a category to a 1-manifold, etc.) and see that the lower level is controlled by quantum groups.
Reshetikhin-Turaev invariants

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They label each component of the knot with a representation, and choose a projection of the knot. They then use the theory of quantum groups to attach maps to small diagrams like:

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These are called the **braiding**, the **quantum trace** and the **coevaluation**.

Composing these together for a given link results in a scalar: the **Reshetikhin-Turaev invariant** for that labeling.
A historical interlude

Progress has been made on categorifying these in a piecemeal fashion for a while.

Khovanov ('99): Jones polynomial ($C_2$ for $sl_2$).

Oszvath-Szabo, Rasmussen ('02): Alexander polynomial (which is actually a $gl(1|1)$ invariant, and doesn't fit into our general picture).

Khovanov ('03): $C_3$ for $sl_3$.

Khovanov-Rozansky ('04): $C_n$ for $sl_n$.

Stroppel-Mazorchuk, Sussan ('06-'07): $\wedge^i C_n$ for $sl_n$.

Cautis-Kamnitzer ('06): $\wedge^i C_n$ for $sl_n$.

Khovanov-Rozansky('06): $C_n$ for $so_n$.

What I want to show is a unified, pictorial construction that should include all of these.
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What I want to show is a unified, pictorial construction that should include all of these. p=proven, c=conjectured.
For our purposes, decategorification means sending

- a vector space $V \rightarrow$ its dimension $\dim V$
- a graded vector space $V \rightarrow$ its $q$-dimension $\dim_q V$
- an abelian category $C \rightarrow$ its Grothendieck group $K_0(C)$
- a graded abelian category $C \rightarrow$ its $q$-Grothendieck group $\mathbb{Z}[q, q^{-1}] \otimes K_0(C)$
- An exact functor $F: C \rightarrow C' \rightarrow$ the induced map $\left[ F \right]: K_0(C) \rightarrow K_0(C')$.

So "$F: C \rightarrow C'$ categorifies $\phi: V \rightarrow V'$" means "there are isomorphisms $K_0(C) \cong V$ and $K_0(C') \cong V'$, such that the map induced by $F$ is $\phi$."

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Decategorification?

For our purposes, decategorification means sending

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So “\( F : C \rightarrow C' \) categorifies \( \phi : V \rightarrow V' \)” means “there are isomorphisms \( K^0(C) \cong V \) and \( K^0(C') \cong V' \), such that the map induced by \( F \) is \( \phi \)”
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In particular, what we’d like to find is

- graded categories $\mathcal{V}^{\lambda_1, \cdots, \lambda_n}$ such that

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K_0^q(\mathcal{V}^{\lambda_1, \cdots, \lambda_n}) \cong V_\lambda = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}
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where $V_{\lambda_i}$ is the representation of $U_q(\mathfrak{g})$ of highest weight $\lambda$. 

Reshetikhin-Turaev invariants
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- Graded functors categorifying:
  - the Chevalley generators $E_i, F_i$ of $U_q(\mathfrak{g})$.
  - the braiding maps relating different orderings of the highest weights.
  - the coevaluation and quantum trace maps.
One of the most successful programs of categorification has been the understanding of quantum groups, which goes back to Lusztig. Some remarkable progress on this story was made in the 80’s and 90’s. Using categorifications:

- Kazhdan and Lusztig defined a basis of the Hecke algebra.
- Lusztig defined the canonical basis of $U_q(\mathfrak{g})$ and its representations.

but this work at its heart was all geometric; there were a lot of perverse sheaves involved.
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but this work at its heart was all geometric; there were a lot of perverse sheaves involved.

While geometry has a lot of power, it’s also kinda hard. Luckily, in the past few years Rouquier and Khovanov-Lauda were able to redigest this whole story combinatorially, and so I can tell you an entirely pictorial story (though there’s a still a little geometry tucked away in corners).
Now we define an algebra $R(\mathfrak{g})$ generated by pictures consisting of strands each colored with a simple root, each labeled with any number of dots with the restrictions that

- strands must begin on $y = 0$, end on $y = 1$
- strands can never be horizontal

Product is given by stacking (and is 0 if ends don’t match).

Let $\mathfrak{g} \cong \mathfrak{sl}_3$ with simple roots $\alpha_1 = (1, -1, 0)$ and $\alpha_2 = (0, 1, -1)$.
The Khovanov-Lauda relations

The relations for $\mathfrak{g} \cong \mathfrak{sl}_3$ are given by (keeping in mind there is an automorphism interchanging blue and green).
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\[
\begin{align*}
\times & = \times + \ | | \\
\times & = \times
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\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\fill[blue] (0,0) circle (3pt);
\draw[blue, thick, -] (-1,-1) -- (1,1);
\draw[blue, thick, -] (1,-1) -- (-1,1);
\end{tikzpicture}
\end{array}
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\begin{array}{c}
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\fill[green] (0,0) circle (3pt);
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\begin{array}{c}
\begin{tikzpicture}
\fill[blue] (0,0) circle (3pt);
\draw[blue, thick, -] (0,0) .. controls (-2,-2) and (-2,0) .. (1,1);
\draw[blue, thick, -] (1,0) .. controls (2,2) and (2,0) .. (0,0);
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& = 0 \\
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There’s a natural ring map $R \otimes R \to R$ given by “horizontal composition”: placing diagrams side by side. Extension of scalars under this map defines a monoidal structure on the category $\mathcal{V}_\infty$ of graded $R$-modules.

**Theorem (Khovanov-Lauda)**
The category $\mathcal{V}_\infty$ categorifies $U_q^+(g)$.

**Theorem (Vasserot-Varagnolo)**
For $g$ simply laced, the category of graded modules over $R(g)$ is derived equivalent to Lusztig’s categorification of $U_q(g)$ and the canonical basis is categorified by the indecomposable projective modules.
Pictorial categorification

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**Theorem (Khovanov-Lauda)**

*The category $\mathcal{V}^\infty$ categorifies $U_q^+(\mathfrak{g})$.***
There’s a natural ring map \( R \otimes R \to R \) given by “horizontal composition”: placing diagrams side by side. Extension of scalars under this map defines a monoidal structure on the category \( \mathcal{V}^\infty \) of graded \( R \)-modules.

**Theorem (Khovanov-Lauda)**

The category \( \mathcal{V}^\infty \) categorifies \( U_+^q(\mathfrak{g}) \).

In fact, this is a combinatorial version of Lusztig’s construction:

**Theorem (Vasserot-Varagnolo)**

For \( \mathfrak{g} \) simply laced, the category of graded modules over \( R(\mathfrak{g}) \) is derived equivalent to Lusztig’s categorification of \( U_+^q(\mathfrak{g}) \) and the canonical basis is categorified by the indecomposable projective modules.
Irreducible representations

Fix a highest weight $\lambda$ of $\mathfrak{g}$. We want a module category over $R(\mathfrak{g})$-mod generated by a “highest weight object” which I draw as a red line. Thus, if I act “horizontally” with $R(\mathfrak{g})$, I’ll get pictures like

But I need to impose relations to get a finite dimensional Grothendieck group:

$$\lambda \cdot \lambda = \lambda \cdot \alpha_1^\vee(\lambda) \cdots = 0$$
$$\lambda \cdot \lambda = \lambda \cdot \alpha_2^\vee(\lambda) \cdots = 0$$
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Theorem (Lauda-Vazirani)

The category $\mathfrak{Y}^\lambda$ with its $\mathfrak{Y}^\infty$ action categorifies the irreducible representation $V_\lambda$. The canonical basis is categorified by the indecomposable projective modules.
Tensor products

Now, some of you might think: “Wait, why is looking at the category of 2-representations hard? Can’t you just take tensor product of the categories?”
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There are a host of reasons why this is a bad idea. For one,

the whole point of quantum groups is that they treat the two sides of the tensor product inequitably. We shouldn’t expect a “democratic” construction, but one slanted toward one tensor factor or another.
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the whole point of quantum groups is that they treat the two sides of the tensor product inequitably. We shouldn’t expect a “democratic” construction, but one slanted toward one tensor factor or another.

Also, the canonical bases give us hints of the structure of the categorifications of things, and the canonical basis of the tensor product is not the tensor product of canonical bases.

In fact, we’ll see that there are objects categorifying the tensor product of the canonical bases, but they are not projective.
Now, some of you might think: “Wait, why is looking at the category of 2-representations hard? Can’t you just take tensor product of the categories?” There are a host of reasons why this is a bad idea. For one, 

the whole point of quantum groups is that they treat the two sides of the tensor product inequitably. We shouldn’t expect a “democratic” construction, but one slanted toward one tensor factor or another.

Also, the canonical bases give us hints of the structure of the categorifications of things, and the canonical basis of the tensor product is not the tensor product of canonical bases. In fact, we’ll see that there are objects categorifying the tensor product of the canonical bases, but they are not projective.
Tensor product algebras

Now we define an algebra generated by pictures consisting of

- red strands, colored with an $\mathfrak{g}$-rep
- non-red strands, colored with simple roots, each labeled with any number of dots
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- strands must begin on $y = 0$, end on $y = 1$ and can never be horizontal
- red strands can never cross

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\[ \sum_{a+b=\alpha^\gamma_1(\lambda)-1} a \cdot b = \lambda \cdot \alpha^{\gamma_1}(\lambda) - 1 \]

Any diagram where a blue or green strand is to the left of all red strands is 0.
Tensor product algebras

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\[
\begin{align*}
\lambda & = \sum_{a+b=\alpha_i^\vee(\lambda)-1} a \cdot \alpha_i^\vee(\lambda) - 1 \cdot b \\
\alpha_1^\vee(\lambda) & = \lambda \\
\alpha_2^\vee(\lambda) & = \lambda
\end{align*}
\]

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First, we impose the Khovanov-Lauda relations from before. Also, we also need some relations involving red lines.

\[
\begin{align*}
\lambda_{a+b=\alpha_1^\vee(\lambda)-1} + \sum_{a+b=\alpha_1^\vee(\lambda)-1} a \cdot b &= \lambda_{\alpha_2^\vee(\lambda)} \\
\lambda &= \alpha_1^\vee(\lambda) \\
\lambda &= \alpha_2^\vee(\lambda)
\end{align*}
\]

Any diagram where a blue or green strand is to the left of all red strands is 0.
If there’s only red line, then we only get one new interesting relation:

\[ \lambda = \lambda \vee 2(\lambda) \cdots = 0 \]

If there’s only one red line labeled with \( \lambda \), then we just get back the category for a simple representation.
Tensor product algebras

For a sequence of representations \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \), let \( E^\lambda \) be the subalgebra where the red lines are labeled with \( \lambda \) in order.

**Theorem**

\[ K(E^\lambda \text{-mod}) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell} = V_\lambda. \]
Tensor product algebras

For a sequence of representations $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $E^\lambda$ be the subalgebra where the red lines are labeled with $\lambda$ in order.

**Theorem**

$$K(E^\lambda \text{-mod}) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell} = V_\lambda.$$  

We can also describe the action of $U_q^-(\mathfrak{g})$ on $V_\lambda$ using this categorification:

**Theorem**

*The “horizontal” action of $R(\mathfrak{g}) \text{-mod}$ induces the usual action of $U_q^-(\mathfrak{g})$ on $V_\lambda$.***

This is the first sanity check: the grading shifts necessary to get to quantum coproduct are “built in.”
Standard modules

The proof is by constructing a set of modules which categorify pure tensors. We call these **standard modules**. Consider the right ideal generated by all pictures where all red/black crossings are “negative.”

![negative crossing](image)

![positive crossing](image)

Let \( S^\lambda \) denote the right \( E^\lambda \)-module given by quotient by this ideal.
Standard modules

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Let $S^\lambda$ denote the right $E^\lambda$-module given by quotient by this ideal.

**Proposition**

$\text{End}(S^\lambda) \cong E^{\lambda_1} \otimes \cdots \otimes E^{\lambda_\ell}$.

So, the failure of $S^\lambda$ to be projective is exactly what encodes the difference between our tensor product category, and the naive tensor product.
We also obtain a functor from the naive product category to our category given by
\[- \bigotimes_{E^\lambda_1 \otimes \cdots \otimes E^\lambda_\ell} S^\lambda : \mathcal{V}^{\lambda_1; \cdots; \lambda_\ell} \to \mathcal{V}^\lambda.\]

**Proposition**

This functor induces $V_\lambda \cong K(\mathcal{V}^{\lambda_1}) \otimes \cdots \otimes K(\mathcal{V}^{\lambda_\ell}) \cong K(\mathcal{V}^\lambda)$. 
Standard modules

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**Proposition**

This functor induces \( V_{\lambda} \cong K(\mathcal{V}^{\lambda_1}) \otimes \cdots \otimes K(\mathcal{V}^{\lambda_\ell}) \cong K(\mathcal{V}^\lambda) \).

To see why this is so, consider \( \mathfrak{F}_i(S^\lambda) \). This has a filtration given by elements whose successive quotients match the terms of the coproduct

\[
\Delta^{(\ell)}(F) = 1 \otimes \cdots \otimes F_i + \cdots + F_i \otimes K_i^{-1} \otimes \cdots \otimes K_i^{-1}.
\]
So, now we need to look for braiding functors.
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Theorem

The derived tensor product $- \otimes L^{\mathcal{B}_i} : \mathcal{V}^\lambda \to \mathcal{V}^{(i,i+1)\cdot \lambda}$ categorifies the braiding map $R_i : \mathcal{V}_\lambda \to \mathcal{V}_{(i,i+1)\cdot \lambda}$. The inverse functor is given by $\text{RHom}(\mathcal{B}_i, -)$.
Coevaluation and quantum trace

We also need functors corresponding to the cups and caps in our theory. First, consider the case where we have two highest weights \( \lambda \) and \(-w_0\lambda = \lambda^*\). In this case, pick a reduced expression

\[
w_0 = s_1 \cdots s_n \text{ with corresponding roots } \alpha_1, \cdots, \alpha_n.\]
Coevolution and quantum trace

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There’s a unique simple module $L_\lambda$ not killed by the idempotent for the sequence of weights and roots $\lambda, \alpha_1^{\vee}(\lambda), \alpha_2^{\vee}(s_1\lambda), \ldots, \alpha_n^{\vee}(s_{n-1}\cdots s_1\lambda), \lambda^*$. 
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There’s a unique simple module $L_{\lambda}$ not killed by the idempotent for the sequence of weights and roots $\lambda, \alpha_1^{(\alpha_1^\vee(\lambda))}, \alpha_2^{(\alpha_2^\vee(s_1 \lambda))}, \ldots, \alpha_n^{(\alpha_n^\vee(s_{n-1} \cdots s_1 \lambda))}, \lambda^*$.

- The coevaluation functor is categorified by the functor $\mathcal{V}^\emptyset \cong \text{Vect} \rightarrow \mathcal{V}^{\lambda, \lambda^*}$ sending $\mathbb{C} \rightarrow L_{\lambda}$.
- The quantum trace functor is categorified by

$$\text{RHom}(L_{\lambda}, -)[2\rho^\vee(\lambda)](2\langle \lambda, \rho \rangle) : \mathcal{V}^{\lambda, \lambda^*} \rightarrow \mathcal{V}^\emptyset \cong D_{\text{fd}}(\text{Vect})$$.
Coevolution and quantum trace

In particular, the algebra (which is the invariant of the circle)

$$A_\lambda = \text{Ext}^\bullet(L_\lambda, L_\lambda)[2\rho^\vee(\lambda)](2\langle\lambda, \rho\rangle)$$

has graded dimension given by the quantum dimension of $V_\lambda$. 

Has anyone seen this algebra before? One interesting candidate is the algebra structure that Feigin, Frenkel and Rybnikov put on $V_\lambda$ using the "quantum shift of argument algebra" at a principal nilpotent. Another tantalizing possibility is that it is related to the geometry of $\text{Gr}_\lambda$. Perhaps a ring structure on intersection cohomology?

Ben Webster (MIT/Oregon)
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Coevolution and quantum trace

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\[
L_{\mu} = \mu \parallel \begin{array}{c}
\lambda_1 \\
\mu \\
\lambda_1 \\
\lambda_\ell
\end{array} \cong \begin{array}{c}
\alpha_2^\vee(\mu) \\
\alpha_1^\vee + \alpha_2^\vee(\mu) \\
\alpha_1^\vee(\mu) \\
\mu^*
\end{array} \parallel \begin{array}{c}
\lambda_\ell \\
\lambda_\ell
\end{array}
\]
Coevolution and quantum trace

To do this in general, you can construct natural bimodules $\mathcal{K}_\lambda$. Rather than give a definition, let me just draw the picture.

Let $\lambda_1, \mu, \lambda_\ell$ be representations.

\[ L_\mu \]

Theorem

Tensor product with this bimodule categorifies coevaluation, and Hom with it categorifies quantum trace.
Now, I should have drawn all these pictures as ribbon knots, since framing matters in our picture. Moreover, I need to associate an actual functor to the ribbon twist.

\[
\text{Ribbon functor is just } \mathcal{M} \mapsto \mathcal{M}(\langle 2\lambda, \rho \rangle)_2 \bigvee (\lambda)^\rho.
\]

Note: this is a strange ribbon element! (It appeared in work of Snyder and Tingley on half-twist elements.) But that won't change things very much.
Ribbon structure

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Note: this is a strange ribbon element! (It appeared in work of Snyder and Tingley on half-twist elements.) But that won’t change things very much.
Now, we start with a picture of our knot (in red), cut it up into these elementary pieces, and compose these functors in the order the elementary pieces fit together.
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For a link $L$, we get a functor $F_L : \mathcal{V}^\emptyset \cong D_{fd}(\text{Vect}) \to \mathcal{V}^\emptyset \cong D_{fd}(\text{Vect})$. So $F_L(\mathbb{C})$ is a complex of vector spaces (actually graded vector spaces).

**Theorem**

The cohomology of $F_L(\mathbb{C})$ is a knot invariant. The graded Euler characteristic of this complex is $J_{V,L}(q)$. 
Knot invariants

Start with \( \mathbb{C} \).

\[ A_1 = \mathbb{C} \otimes K_1, \]
\[ A_2 = B_1 \otimes K_1, \]
\[ A_3 = \text{RHom}(B_i, B_2), \]
\[ A_4 = B_3 \otimes B_1, \]
\[ A_5 = B_4 \otimes B_3, \]
\[ A_6 = \text{RHom}(K_2, B_5), \]
\[ A_7 = \text{RHom}(K_1, B_6). \]
Knot invariants

\[ V \otimes V^* \]

\[ A_1 = \mathbb{C} \otimes \mathcal{H}_V^{1,2} \]

Replace with projective resolution \( B_1 \)

Start with \( \mathbb{C} \).
Knot invariants

\[ V \cong V^* \]

\[ A_2 = B_1 \otimes \mathcal{H}_V^{1,2} \]

Replace with injective resolution \( B_2 \)

Replace with projective resolution \( B_1 \)

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Start with \( \mathbb{C} \).
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$A_1 = \mathbb{C} \otimes \mathcal{H}_V^{1,2}$

Replace with projective resolution $B_1$

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Replace with injective resolution $B_2$

$A_3 = \text{RHom}(\mathcal{B}_i, B_2)$

Replace with projective resolution $B_3$

$A_4 = B_3 \otimes \mathcal{B}_1$

Replace with projective resolution $B_4$
Start with $\mathbb{C}$.

\begin{align*}
A_1 &= \mathbb{C} \otimes \mathcal{R}_V^{1,2} \\
A_2 &= B_1 \otimes \mathcal{R}_V^{1,2} \\
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A_4 &= B_3 \otimes \mathcal{B}_1 \\
A_5 &= B_4 \otimes \mathcal{B}_3 \\
\end{align*}

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Start with \( \mathbb{C} \).

Replace with projective resolution \( B_1 \).

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Replace with injective resolution \( B_6 \).
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\text{Knot homology!} \\
\text{Replace with projective resolution } B_1 \\
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\text{Replace with injective resolution } B_2 \\
\text{Replace with injective resolution } B_4 \\
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Functoriality?

It’s not known at the moment if this is functorial in cobordisms between knots. How would one construct a functoriality map?

Cobordisms of knots can be cut (using a Morse function) into the moves of

circle destruction

\[
\begin{array}{c}
\text{circle destruction} \\
\begin{array}{c}
\text{\rightarrow} \\
\emptyset
\end{array}
\end{array}
\]

tsaddle move

\[
\begin{array}{c}
\begin{array}{c}
\text{\rightarrow} \\
\text{\saddle}
\end{array}
\end{array}
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circle creation

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\begin{array}{c}
\begin{array}{c}
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Being able to define these maps requires that the cap and cup functors are biadjoint (they’re clearly adjoint one way). However, one has to prove that this map does not depend on the handle decomposition. Not easy!
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Cobordisms of knots can be cut (using a Morse function) into the moves of

- circle destruction: \( \textcircled{\bullet} \rightarrow \emptyset \)
- saddle move: \( ( \rightarrow \quad ) \)
- circle creation: \( \emptyset \rightarrow \textcircled{\bullet} \)

Being able to define these maps requires that the cap and cup functors are biadjoint (they’re clearly adjoint one way).

However, one has to prove that this map does not depend on the handle decomposition. Not easy!
Open questions

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What about Alexander polynomial? Could this prescription be modified to give Knot Floer homology?

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Thanks, y’all.