Mirković-Vilonen polytopes and Khovanov-Lauda-Rouquier algebras

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Abstract. We describe how Mirković-Vilonen polytopes arise naturally from the categorification of Lie algebras using Khovanov-Lauda-Rouquier algebras. This gives an explicit description of the unique crystal isomorphism between simple representations of the KLR algebra and MV polytopes.

MV polytopes as defined from the geometry of the affine Grassmannian only make sense for finite dimensional semi-simple Lie algebras, but our construction actually gives a map from the infinity crystal to polytopes in all symmetrizable Kac-Moody algebras. However, to make the map injective and have well-defined crystal operators on the image, we must in general decorate the polytopes with some extra information. We suggest that the resulting KLR polytopes are the general-type analogues of MV polytopes.

We give a combinatorial description of the resulting decorated polytopes in all affine cases, and show that this recovers the affine MV polytopes recently defined by Kamnitzer and Baumann and the first author in symmetric affine types. We also briefly discuss the situation beyond affine type.

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Introduction

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. In recent years, a number of parametrizations of the crystal $B(-\infty)$ for $U_q^+(\mathfrak{g})$ have been studied. In this paper, we consider the relationship between two of these. In the first, the indexing set is the set $MV$ of Mirković-Vilonen polytopes, as introduced by Anderson [And03] and studied by Kamnitzer [Kam10, Kam07], building on Mirković and Vilonen’s work [MV07] on the geometry of the affine Grassmannian. In the second, the indexing set is the set $KLR$ of simple gradable modules of Khovanov-Lauda-Rouquier (KLR) algebras, as developed by Lauda-Vazirani [LV11] and Kleshchev-Ram [KR]. Since both of these sets index $B(-\infty)$, there is a unique crystal isomorphism between them, but this bijection has not previously been described directly.

Here we give a simple description of this bijection: There is a KLR algebra $R(\nu)$ attached to each positive sum $\nu = \sum a_i \alpha_i$ of simple roots. For any two such weights $\nu_1, \nu_2$, there is a natural inclusion $R(\nu_1) \otimes R(\nu_2) \hookrightarrow R(\nu_1 + \nu_2)$. Let $P_L$ be the convex hull of the weights $\nu'$ such that $\text{Res}^{\nu'}_{\nu, \nu-\nu'} L \neq 0$.

**Theorem A**  The map which sends a simple $R(\nu)$-module $L$ to $P_L$ is the unique crystal isomorphism between $KLR$ and $MV$.

We feel Theorem A is interesting in its own right, but perhaps more important is the fact that $KLR$ naturally indexes $B(-\infty)$ for any symmetrizable Kac-Moody algebra. Thus, one can try to use the map above to define Mirković-Vilonen polytopes outside of finite type. However, one finds pairs of non-isomorphic simples with the same polytopes (for example, this occurs in $\widehat{sl}_2$ in the weight space $4\delta$), so the polytopes alone are not enough information to parametrize $B(-\infty)$.

As suggested by Dunlap [Dun10] and developed in [BKT], this problem can be overcome by decorating the edges of $P_L$ with extra information. In the current setting, the most natural data to associate to an edge is a semi-cuspidal representation of a smaller KLR algebra (see Definition 2.2). In complete generality, there are many different semi-cuspidal representations that can decorate a given edge, and we do not know a fully combinatorial description of the resulting object.

For edges parallel to real roots there is only one possible semi-cuspidal representation, so it is safe to leave off the decoration. In particular, in finite type one can ignore the decoration altogether, and the polytopes are described by Theorem A. The next
simplest types are the affine algebras. There the decoration does play an important role, but we nonetheless obtain a combinatorial object.

For now restrict to the case when \( g \) is affine of rank \( r + 1 \). Then \( g \) has only one minimal imaginary root \( \delta \), and this has multiplicity \( r \). It turns out that the semicuspidal representations that can be associated to a given edge of \( P_L \) parallel to \( \delta \) are naturally indexed by an \( r \)-tuple of partitions (see Lemma 3.31). In fact, we can reduce the amount of information even further: as in [BKT], the (possibly degenerate) \( r \)-faces of \( P_L \) parallel to \( \delta \) are naturally indexed by the chamber coweights \( \gamma \) of an underlying finite type root system. Denote the face of \( P_L \) corresponding to \( \gamma \) by \( P^\gamma_L \). We in fact decorate \( P_L \) with just the data of a partition \( \pi_{\gamma} \) for each chamber coweight \( \gamma \) (see Definition 3.32) in such a way that, for any edge \( E \) parallel to \( \delta \),

\[
E \quad \text{is a translate of} \quad \left( \sum_{E \subset P^\gamma_L} |\pi_{\gamma}| \right) \delta.
\]

The representation attached to such an edge \( E \) is determined in a natural way by \( \{\pi_{\gamma} : E \subset P^\gamma_L\} \).

Define a **decorated affine pseudo-Weyl polytope** to be a pair consisting of
- a polytope \( P \) in the root lattice of \( g \) with all edges parallel to roots, and
- a choice of partition \( \pi_{\gamma} \) for each chamber coweight \( \gamma \) of the underlying finite type root system which satisfies condition (1) for each edge parallel to \( \delta \).

Let \( P^{KLR} \) be the set of such polytopes which arrive as \( P_L \) for some \( L \) (which we call KLR-polytopes). As in finite type, we seek a combinatorial characterization of \( P^{KLR} \).

Notice that for every 2-face \( F \) of a decorated pseudo-Weyl polytope, the roots parallel to \( F \) form a rank 2 sub-root system \( \Delta_F \) of either finite or affine type. If \( \Delta_F \) is of affine type, then \( F \) has two edges parallel to \( \delta \), which are of the form \( E_{\gamma} = F \cap P_{\gamma} \) and \( E_{\gamma'} = F \cap P_{\gamma'} \) for unique chamber coweights \( \gamma, \gamma' \). We would like to decorate these imaginary edges with \( \pi_{\gamma} \) and \( \pi_{\gamma'} \), but this fails to satisfy (1) since \( E_{\gamma} \) and \( E_{\gamma'} \) are too long. Instead, \( F \) is the Minkowski sum of the line segment \( \left( \sum_{\gamma:F \subset P_{\xi}} |\pi_{\xi}| \right) \delta \) with a decorated pseudo-Weyl polytope \( \tilde{F} \), obtained by shortening \( E_{\gamma} \) and \( E_{\gamma'} \) and decorating them with \( \pi_{\gamma} \) and \( \pi_{\gamma'} \). We will show that:

**Theorem B** For \( g \) an affine Lie algebra, the polytopes \( P_L \) are precisely the decorated pseudo-Weyl polytopes where every 2-dimensional face \( F \) satisfies
- If \( \Delta_F \) is a finite type root system, then \( F \) is an MV polytope for that root system (i.e. it satisfies the tropical Plücker relations from [Kam10]).
- If \( \Delta_F \) is of affine type, then \( \tilde{F} \) is an MV polytope for that rank 2 affine algebra (either \( \widehat{sl}_2 \) or \( A^{(2)}_2 \)) as defined in [BDKT].

The description of rank 2 affine MV polytopes in [BDKT] is combinatorial, so Theorem B gives a combinatorial characterization of KLR polytopes in all affine cases.
In [BKT] analogues of MV polytopes were constructed in all symmetric affine types as decorated Harder-Narasimhan polytopes, and it was shown that these are characterized by their 2-faces. Thus Theorem B also allows us to understand the relationship between our decorated polytopes and those defined in [BKT]:

**Theorem C** Assume \( \mathfrak{g} \) is of affine type with symmetric Cartan matrix. Fix \( b \in B(-\infty) \) and let \( L \) be the corresponding element of \( \mathcal{KLR} \). Our polytope \( P_L \) and the decorated Harder-Narasimhan polytope \( HN_b \) from [BKT] have identical underlying polytopes. Furthermore, for each chamber coweight \( \gamma \) in the underlying finite type root system, the partition \( \lambda_\gamma \) decorating \( HN_b \) as defined in [BKT, Sections 1.5 and 7.6] is the transpose of our \( \pi_\gamma \).

It is natural to ask for an intrinsic characterization of the polytopes \( P_L \) in the general Kac-Moody case. We do not even have a conjecture for a true combinatorial characterization, since the polytopes are decorated with various semi-cuspidal representations, which at the moment are not well-understood. Some difficulties that come up outside of a finite type are discussed in Section 3.6. However, our construction does still satisfy the most basic properties one would expect, as we now summarize (see Corollaries 3.9 and 3.10 for precise statements).

**Theorem D** For \( \mathfrak{g} \) an arbitrary symmetrizable Kac-Moody algebra, the map from \( \mathcal{KLR} \) to polytopes with edges labeled by semi-cuspidal representations is injective. Furthermore, for each convex order on roots, the elements of \( \mathcal{KLR} \) are parameterized by the tuples of representations of smaller KLR algebras decorating the edges along a corresponding path through the polytope, generalizing the parameterization of crystals in finite type by Lusztig data.

As we were completing this paper, some independent work on similar problems appeared: McNamara [McN] proved a version of Theorem D in finite type (amongst other theorems on the structure of these representations) and Kleshchev [Kle] gave a generalization of this to affine type. While there was some overlap with the present paper, these other works were focused on a single convex order, rather than giving a description of how different orders interact as we do in Theorems A, B and C.

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1. Background

1.1. Crystals. Fix a symmetrizable Kac-Moody algebra and let $\Gamma = (I, E)$ be its Dynkin diagram. We are interested in the crystal $B(-\infty)$ associated with $U^+(g)$. This is a combinatorial object arising from the theory of crystal bases for the corresponding quantum group (see e.g. [Kas95]). This section contains a brief explanation of the results we need, roughly following [Kas95] and [HK02], to which we refer the reader for details. We start with a combinatorial notion of crystal that includes many examples which do not arise from representations, but which is easy to characterize combinatorially.

**Definition 1.1** (see [Kas95, Section 7.2]) A **combinatorial crystal** is a set $B$ along with functions $\text{wt}: B \rightarrow P$ (where $P$ is the weight lattice), and, for each $i \in I$, $\varepsilon_i, \varphi_i: B \rightarrow \mathbb{Z} \cup \{-\infty\}$ and $e_i, f_i: B \rightarrow B \sqcup \{\emptyset\}$, such that

(i) $\varphi_i(b) = \varepsilon_i(b) + \langle \text{wt}(b), \alpha_i^\vee \rangle$.

(ii) $e_i$ increases $\varphi_i$ by 1, decreases $\varepsilon_i$ by 1 and increases $\text{wt}$ by $\alpha_i$.

(iii) $f_ib = b'$ if and only if $e_ib' = b$.

(iv) If $\varphi_i(b) = -\infty$, then $e_ib = f_ib = \emptyset$.

We often denote a combinatorial crystal simply by $B$, suppressing the other data.

**Definition 1.2** A **lowest weight** combinatorial crystal is a combinatorial crystal which has a distinguished element $b_-$ (the lowest weight element) such that

(i) The lowest weight element $b_-$ can be reached from any $b \in B$ by applying a sequence of $f_i$ for various $i \in I$.

(ii) For all $b \in B$ and all $i \in I$, $\varphi_i(b) = \max \{n : f_i^n(b) \neq \emptyset\}$.

Notice that, for a lowest weight combinatorial crystal, the functions $\varphi_i, \varepsilon_i$ and $\text{wt}$ are determined by the $f_i$ and the weight $\text{wt}(b_-)$ of just the lowest weight element.

It will be convenient for us to consider a slightly stronger notion, which is less common in the literature:

**Definition 1.3** A **bicrystal** is a set $B$ with 2 different crystal structures whose weight functions agree. We will always use the convention of placing a star superscript on all data for the second crystal structure, so $e_i^*, f_i^*, \varphi_i^*$, etc.

We will consider one very important example of a bicrystal: the universal lowest weight crystal $B(-\infty)$ along with the usual crystal operators and Kashiwara’s $*$-crystal operators, which are the conjugates $e_i^* = *e_i, f_i^* = *f_i$ of the usual operators by Kashiwara’s involution $*: B(-\infty) \rightarrow B(-\infty)$ (see [Kas93, 2.1.1]). The involution $*$ is a crystal limit of the bar involution of $U^+_q(g)$, but it also has a simple combinatorial definition in each of the models of interest to us.
The following is a rewording of [KS97, Proposition 3.2.3] designed to make the roles of the usual crystal operators and the \(*\)-crystal operators more symmetric:

**Proposition 1.4** Fix a bicrystal \(B\). Assume \((B, e_i, f_i)\) and \((B, e'_i, f'_i)\) are both lowest weight combinatorial crystals with same the same lowest weight element \(b_-\), where the other data is determined by setting \(\text{wt}(b_-) = 0\). Assume further that, for all \(i \neq j \in I\) and all \(b \in B\),

(i) \(e_i(b), e'_i(b) \neq 0\).

(ii) \(e'_i e_i(b) = e_i e'_i(b)\).

(iii) For all \(b \in B\), \(\varphi_i(b) = \varphi'_i(b) - \text{wt}(b)\alpha'_i \geq 0\)

(iv) If \(\varphi_i(b) + \varphi'_i(b) - \text{wt}(b)\alpha'_i = 0\) then \(e_i(b) = e'_i(b)\).

(v) If \(\varphi_i(b) + \varphi'_i(b) - \text{wt}(b)\alpha'_i \geq 1\) then \(\varphi'_i(e_i(b)) = \varphi'_i(b)\) and \(\varphi_i(e'_i(b)) = \varphi_i(b)\).

(vi) If \(\varphi_i(b) + \varphi'_i(b) - \text{wt}(b)\alpha'_i \geq 2\) then \(e_i e'_i(b) = e'_i e_i(b)\).

then \((B, e_i, f_i) \simeq (B, e'_i, f'_i) \simeq B(-\infty)\), and \(e'_i = *e_i*\, f'_i = *f_i*\), where \(*\) is Kashiwara’s involution. Furthermore, these conditions are always satisfied by \(B(-\infty)\) along with its operators \(e_i, f_i, e'_i, f'_i\).

**Proof.** We simply explain how [KS97, Proposition 3.2.3] implies our statement, referring the reader there for specialized notation. Define the map

\[
B \rightarrow B \otimes B_i
\]

\[
b \mapsto (f'_i)^{\varphi_i(b)}(b) \otimes e_i^{\varphi'_i(b)}b_i.
\]

Then one can check that our conditions imply all the conditions from [KS97, Proposition 3.2.3], so that result implies the crystal structure on \(B\) defined by \(e_i, f_i\) is isomorphic to \(B(-\infty)\). The remaining statements then follow from [KS97, Theorem 3.2.2]. \(\square\)

We will also make use of Saito’s crystal reflections from [Sai94].

**Definition 1.5** Fix \(b \in B(-\infty)\) with \(\varphi'_i(b) = 0\). The Saito reflection of \(b\) is \(\sigma_i b = (e_i^{\varphi}_i)^{e_i(b)} f_i^{\varphi_i(b)} b\). There is also a dual notion of Saito reflection defined by \(\sigma'_i b := *(\sigma_i(*b))\) which is defined for those \(b\) such that \(\varphi_i(b) = 0\)

The operation \(\sigma_i\) does in fact reflect the weight of \(b\) by \(s_i\), as the name suggests.

1.2. **Convex orders and charges.** Fix a symmetrizable Kac-Moody algebra \(\mathfrak{g}\) with root system \(\Delta\) and Cartan subalgebra \(\mathfrak{h}\). Let \(\Delta^{\text{min}}_+\) be the set of positive roots \(\alpha\) such that \(x\alpha\) is not a root for any \(0 < x < 1\) (this is all positive roots in finite type).

**Definition 1.6** A convex (pre)order is a (pre)order on \(\Delta^{\text{min}}_+\) such that, given \(S, S' \subset \Delta^{\text{min}}_+\) with \(S \cup S' = \Delta^{\text{irr}}_+\) and \(\alpha > \alpha'\) for all \(\alpha \in S, \alpha' \in S'\), the convex cones \(\text{span}_{\mathbb{R}_{\geq 0}} S\) and \(\text{span}_{\mathbb{R}_{\geq 0}} S'\) intersect only at the origin.
Notice that any preorder on $\Delta^\text{min}_+$ extends to a preorder on all positive roots, where proportional roots are equivalent.

**Definition 1.7** A charge is a linear function $c : h^* \to \mathbb{C}$ such that $c(\alpha_i) \neq 0$ for each simple root $\alpha_i$ and such that all $c(\alpha_i)$’s (and thus the images of all positive roots) lie in the upper half-plane.

Every charge defines a preorder $>_c$ on $\Delta^\text{min}_+$ by setting $\alpha \geq c \beta$ if and only if $\text{arg}(c(\alpha)) \geq \text{arg}(c(\beta))$, where arg is the argument of the complex number (taking a branch cut of log which does not intersect the upper half plane), and this is clearly convex. If $c$ is generic, $>_c$ is a total order. We will need the following notion of “reflection” for convex orders and charges.

**Definition 1.8** Fix a convex order $>$ such that $\alpha_i$ is lowest (resp. greatest). Define a new convex order $>_i$ by

$$\beta > \gamma \iff s_i \beta >_i s_i \gamma \quad \beta, \gamma \neq \alpha_i$$

and $\alpha_i$ greatest (resp. lowest) for $>_i$.

Similarly, for a charge $c$ such that $\text{arg} \alpha_i$ is lowest (resp. greatest) amongst positive roots, define a new charge $c^i$ by $c^i(\nu) = c(s_i(\nu))$. This will not always send $\Delta_+$ to the upper half plane, but it will send it to some half plane, and we can then rotate to make that the upper half plane (in the end we only care about the order on roots, so the precise rotation does not matter).

It should be clear from the definitions that the reflections for charges and convex orders are compatible in the sense that, for all charges $c$ such that $\alpha_i$ is greatest or lowest, $(>^i)_c$ and $>_c$ coincide.

**Definition 1.9** Fix a pair $(\alpha, >)$, where $\alpha$ is a positive root and $>$ is a convex order on $\Delta^\text{min}_+$. A charge $c$ is said to be $(\alpha, >)$ compatible if, for all $\beta \in \Delta_+$ such that $\alpha - \beta \in \text{span}_{\mathbb{Z} \geq 0} \{\alpha_i\}$, we have $\alpha \prec \beta$ if and only if $\alpha < c \beta$ and $\alpha > \beta$ if and only if $\alpha > c \beta$.

For any fixed $\alpha$ there are only finitely many positive roots $\beta$ with $\alpha - \beta \in \text{span}_{\mathbb{Z} \geq 0} \{\alpha_i\}$, so it follows easily from the definition of convex order that, for any pair $(\alpha, >)$, there exists a $(\alpha, >)$-compatible charge. Not all convex orders arise from charges, but in many instances the existence of $(\alpha, >)$-compatible charges will allow us to restrict to convex orders that do.

The following is well known; however, since we have used a slightly unusual definition of convex order, we include a proof for completeness.

**Proposition 1.10** Assume $\mathfrak{g}$ is of finite type. There is a bijection between convex orders on $\Delta_+$ and expressions $i = i_1 \cdots i_N$ for the longest word $w_0$, which is given by sending $i$ to the order $\alpha_{i_1} > s_{i_1} \alpha_{i_2} > s_{i_1}s_{i_2} \alpha_{i_3} > \cdots > s_{i_1} \cdots s_{i_{N-1}} \alpha_{i_N}$.
Proof. First, fix a reduced expression. It is well known that
\[ \{ \alpha_{i_1}, s_i \alpha_{i_2}, s_i s_i \alpha_{i_3}, \ldots, s_i \cdots s_i \alpha_{i_N} \} \]
is an enumerate of the positive roots, so we have defined a total ordering on positive roots. Any pair \( S, S' \) as in Definition 1.6 for this order is of the form
\[ S = \{ \alpha_{i_1}, s_i \alpha_{i_2}, \ldots, s_i \cdots s_i \alpha_{i_b} \}, \quad \text{and} \quad S' = \{ s_i \cdots s_i \alpha_{i_{b+1}}, \ldots, s_i \cdots s_i \alpha_{i_N} \}. \]
The convex cones for these sets are separated by the hyperplane defined by \( s_i \cdots s_i \rho^\vee \), so are clearly disjoint. This proves that all the orders coming from reduced expressions in this way are convex.

Now fix a convex order \( > \). There is a unique greatest root, which must be a simple root \( \alpha_{i_1} \), since otherwise it would be in the span of the other positive roots, contradicting convexity. The convex order \( >^{s_{i_1}} \) as defined above also has a greatest root \( \alpha_{i_2} \). Define \( i_3 \) in the same way using \( >^{s_{i_1} s_{i_2}} \) and continue as many times as there are positive roots. The list \( \alpha_{i_1}, s_i \alpha_{i_2}, s_i s_i \alpha_{i_3}, \ldots, s_i \cdots s_i \alpha_{i_N} \) is a complete, irredundant list of positive roots. This implies that \( i \) is a reduced expression for \( w_0 \). Furthermore, if we apply the procedure in the statement to create an order on positive roots from this expression, we clearly end up with our original convex order. \( \square \)

Of course, if \( \mathfrak{g} \) is of infinite type, applying the technique of this proof will result not in a reduced word for the longest element (which does not exist), but an infinite reduced word \( i_1, i_2, i_3, \ldots \) in \( I \) as well as a dual sequence \( \ldots, i_{-3}, i_{-2}, i_{-1} \) constructed from looking at lowest elements. The corresponding lists of roots
\[ \alpha_{i_1} > s_i \alpha_{i_2} > s_i s_i \alpha_{i_3} > \ldots \quad \text{and} \quad \ldots > s_{i_{-1}} s_{i_{-2}} s_{i_{-3}} > s_{i_{-1}} s_{i_{-2}} > s_{i_{-1}} \]
are totally ordered, but don’t contain every root. In the affine case, only \( \delta \) will be lacking, but for hyperbolic algebras, we can miss many real roots, or even simple roots. In general, the first list will contain all real roots larger than any imaginary root, and the second all such roots smaller than any imaginary root.

Definition 1.11 Fix a convex order \( > \). For each \( b \in B(\infty) \) and each real root \( \alpha \) such that \( \alpha > \beta \) for all imaginary roots \( \beta \), we define an integer \( \alpha^*_\alpha(b) \). This is done inductively by setting \( \alpha_{i_1}^*(b) = \varphi_i(b) \) if \( \alpha_{i_1} \) lowest for \( > \), and
\[ \alpha_{i_2}^*(b) = \alpha_{s_{i_1}}^*(\alpha_{i_2}^*(b)) \]
for all other roots. The collection of such \( \alpha_{i_2}^*(b) \) is called the crystal-theoretic real Lusztig data for \( b \) with respect to \( > \).

Similarly, define the dual crystal theoretic Lusztig data \( \alpha_{i_2}^\dagger \) for all real roots smaller than all imaginaries by switching starred and unstarred operators. For finite type groups, this is just the Lusztig data for the opposite order, but in infinite types it is new information.
Remark 1.12 In finite type, Theorem A, along with results in [Kam10, Kam07], shows that the $a_{\alpha_i}(b)$ are the exponents in Lusztig’s PBW basis element corresponding to $b$ for an appropriate reduced expression of $w_0$. This explains the terminology.

1.3. Pseudo-Weyl polytopes. We use notation as in the previous subsection.

Definition 1.13 A pseudo-Weyl polytope is a convex polytope $P$ in $\mathfrak{h}^*$ with all edges parallel to roots.

Definition 1.14 For a pseudo-Weyl polytope $P$, let $\mu_0(P)$ be the vertex of $P$ such that $\langle \mu_0(P), \rho^\vee \rangle$ is lowest, and $\mu^0(P)$ the vertex where this is highest (these are vertices as for all roots $\langle \alpha, \rho^\vee \rangle$, $0$).

Lemma 1.15 Fix a pseudo-Weyl polytope $P$ and a convex total order $>$ on $\Delta^\min$. There is a unique path $P^>$ through the 1-skeleton of $P$ from $\mu_0(P)$ to $\mu^0(P)$ which passes through at most one edge parallel to each root, and these appear in decreasing order according to $>$. □

Definition 1.16 Fix a pseudo-Weyl polytope $P$ and a convex order $>$. For each $\alpha \in \Delta^\min$, define $a^>_\alpha(P)$ to be the unique non-negative number such that the edge in $P^>$ parallel to $\alpha$ is a translate of $a^>_\alpha(P)\alpha$. We call the collection $\{a^>_\alpha(P)\}$ the geometric Lusztig data of $P$ with respect to $>$.  

Lemma 1.17 Let $P$ be a pseudo-Weyl polytope and $e$ an edge of $P$. Then there exists a charge $c$ such that $>_c$ is a total order and $e \subset P^>_c$. In particular, a pseudo-Weyl polytope $P$ is uniquely determined by its Lusztig data with respect to all convex orders $>_c$ coming from charges.

Proof. Since $e$ is an edge of $P$, there is a functional $\phi \in \mathfrak{h}$ such that

$$e = \{p \in P : \langle p, \phi \rangle \text{ is greatest} \}.$$ 

If $e$ is parallel to the root $\beta$, this means $\langle \beta, \phi \rangle = 0$, and $\phi$ may be chosen so that $\langle \beta', \phi \rangle \neq 0$ for all other $\beta'$ which are parallel to edges of $P$. For any linear function $f : \mathfrak{h} \rightarrow \mathbb{R}$ such that $f(\Delta^+) \subset \mathbb{R}_+$, define a charge $c_f$ by

$$c_f(p) = \phi(p) + f(p)i.$$
For generic $f$, the charge $c_f$ satisfies the required conditions. \hfill $\square$

The following should be thought of as a general-type analogue of the fact that, in finite type, each reduced expression for $w_0$ can be obtained from any other reduced expression by a finite number of braid moves. In fact, this statement can be generalized to include all convex orders, not just those coming from charges, but we only need the simpler version.

**Lemma 1.18** Let $P$ be a Pseudo-Weyl polytope and $c, c'$ two generic charges. Then there is a sequence of generic charges $c_0, c_1, \ldots, c_k$ such that $P > c_0 = P > c$, $P > c_k = P > c'$, and, for all $k \leq j < k$, $P > c_j$ and $P > c_{j+1}$ differ by moving around a single 2-face of $P$ in the two possible directions.

**Proof.** Let $\Delta^{\text{res}}$ be the set of root directions that appear as edges in $P$. For $0 \leq t \leq 1$, let $c_t = (1 - t)c + tc'$. Clearly this is a charge. We can deform $c, c'$ slightly, without changing the order of any of the roots in $\Delta^{\text{res}}$, such that

- For all but finitely many $t$, $c_t$ induces a total order on $\Delta^{\text{res}}$.
- For those $t$ where $c_t$ does not induce a total order, there is exactly one argument $0 < a_t < \pi$ such that more than one root in $\Delta^{\text{res}}$ has argument $a_t$. Furthermore, the span of the roots with argument $a_t$ is 2 dimensional.

Denote the values of $t$ where $c_t$ does not induce a total order by $s_1, \ldots, s_{k-1}$. Fix $t_1, \ldots, t_k$ with

$$0 = t_0 < s_1 < t_1 < s_2 \ldots < t_{k-1} < s_{k-1} < t_k = 1.$$

Then $c_j = c_{t_j}$ is the required sequence. \hfill $\square$

### 1.4. Finite type MV polytopes.

In finite type, Anderson [And03] and Kamnitzer [Kam10, Kam07] developed a realization of $B(-\infty)$ where the underlying set consists of Mirković-Vilonen (MV) polytopes. These are certain polytopes in weight space. Here we will need are the certain characterization theorems, which we now discuss.

**Proposition 1.19** Assume $g$ finite type. There is a unique map $b \to P_b$ from $B(-\infty)$ to pseudo-Weyl polytopes such that

(i) $\text{wt}(b) = \mu_0(P_b) - \mu_0(P_b)$.

(ii) If $>$ is a convex order with minimal root $\alpha_i$, then for all $\beta \neq \alpha_i$, $a^>_p(P_{c(b)}) = a^>_p(P_b)$, and $a^>_i(P_{c(b)}) = a^>_i(P_b) + 1$.

(iii) If $>$ is a convex order with minimal root $\alpha_i$ and $\varphi_i(P_b) = 0$, then for all $\beta \neq \alpha_i$, $a^>_p(P_b) = a^>_{\varphi_i}(P_{c(b)})$ and $a_{\alpha_i}(P_{c(b)}) = 0$.

This map is the unique bicrystal isomorphism between $B(-\infty)$ and the set of MV polytopes. \hfill $\square$

**Proof.** The first step is to show that there is at most one map $b \to P_b$ satisfying the conditions. To see this we proceed by induction. Consider the reverse-lexicographical
order on collections of integers $a = (a_k)_{1 \leq k \leq N}$. Assume $a$ is minimal such that, for some convex order

$$\beta_1 > \beta_2 > \cdots > \beta_N$$

and for two maps $b \to P_b$ and $b \to P'_b$ satisfying the conditions, $a_{\beta_k}(P_b) = a_k$ for all $k$, but $a_{\beta_k}(P'_b) \neq a_k$ for some $k$. If $a_N \neq 0$ we can reduce to a smaller such example using condition (ii). Otherwise, as long as some $a_k \neq 0$, we can reduce to a smaller such example using (iii). Clearly the map is unique if all $a_k = 0$, so this proves uniqueness.

So, it remains to show that $b \to MV_b$ does satisfy both conditions. But this is immediate from [Sai94, Proposition 3.4.7] and the fact that the integers $a_{\beta_k}(MV_b)$ agree with the exponents in the Lusztig’s PBW monomial corresponding to $b$, which is shown in [Kam10, Theorem 7.2].

The following is immediate from 1.19

**Corollary 1.20** In finite type, the geometric Lusztig data for an MV polytope will always agree with the crystal theoretic Lusztig data for the corresponding element of $B(-\infty)$ (see Definition 1.11).

We also need the following standard facts about MV polytopes:

**Theorem 1.21** ([Kam10, Theorem D]) The MV polytopes are exactly those pseudo-Weyl polytopes such that all 2-faces are MV polytopes for the corresponding rank 2 root system.

**Theorem 1.22** ([Kam10, 4.2]) An MV polytope is uniquely determined by its Lusztig data with respect to any one convex order on positive roots.

1.5. **Rank 2 affine MV polytopes.** We briefly review the MV polytopes associated to the affine root systems $\check{sl}_2$ and $A_2^{(2)}$ in [BDKT], and recall a characterization of the resulting polytopes developed in [MT].

The $\check{sl}_2$ and $A_2^{(2)}$ root systems correspond to the affine Dynkin diagrams

$$\check{sl}_2 : \begin{array}{c} \bullet \leftrightarrow \bullet \end{array}, \quad A_2^{(2)} : \begin{array}{c} \bullet \Leftrightarrow \bullet \end{array}.$$

The corresponding symmetrized Cartan matrices are

$$\check{sl}_2 : \quad N = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad A_2^{(2)} : \quad N = \begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}.$$

Denote the simple roots by $\alpha_0, \alpha_1$, where in the case of $A_2^{(2)}$ the short root is $\alpha_0$. Define $\delta = \alpha_0 + \alpha_1$ for $\check{sl}_2$ and $\delta = 2\alpha_0 + \alpha_1$ for $A_2^{(2)}$.

The dual Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$ is a three dimensional vector space containing $\alpha_0, \alpha_1$. This has a standard non-degenerate bilinear form $(\cdot, \cdot)$ such that $(\alpha_i, \alpha_j) = N_{i,j}$. 

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Notice that \((\alpha_0, \delta) = (\alpha_1, \delta) = 0\). Fix fundamental coweights \(\omega_0, \omega_1\) which satisfy \((\alpha_i, \omega_j) = \delta_{i,j}\), where we are identifying coweight space with weight space using \((,\cdot)\).

The set of positive roots for \(\hat{sl}_2\) is

\[
\{\alpha_0, \alpha_0 + \delta, \alpha_0 + 2\delta, \ldots\} \cup \{\alpha_1, \alpha_1 + \delta, \alpha_1 + 2\delta, \ldots\} \cup \{\delta, 2\delta, 3\delta, \ldots\},
\]

where the first two families consist of real roots and the third family of imaginary roots. The set of positive roots for \(A_2^{(2)}\) is

\[
\Delta_{re}^+ = \{\alpha_0 + k\delta, \alpha_1 + 2k\delta, \alpha_0 + \alpha_1 + k\delta, 2\alpha_0 + (2k+1)\delta \mid k \geq 0\}
\]

and

\[
\Delta_{im}^+ = \{k\delta \mid k \geq 1\},
\]

where \(\Delta_{re}^+\) consists of real roots and \(\Delta_{im}^+\) of imaginary roots. We draw these as

\[
\begin{align*}
\sim \hat{sl}_2 & \quad \sim A_2^{(2)} \\
\end{align*}
\]

**Definition 1.23** Label the positive real roots by \(r_k, r^k\) for \(k \in \mathbb{Z}_{>0}\) by:

- For \(\hat{sl}_2\): \(r_k = \alpha_1 + (k-1)\delta\) and \(r^k = \alpha_0 + (k-1)\delta\).
- For \(A_2^{(2)}\):

\[
R_k = \begin{cases} 
\tilde{\alpha}_1 + (k-1)\tilde{\delta} & \text{if } k \text{ is odd,} \\
\tilde{\alpha}_0 + \tilde{\alpha}_1 + \tilde{\delta} & \text{if } k \text{ is even,}
\end{cases} \\
R^k = \begin{cases} 
\tilde{\alpha}_0 + (k-1)\tilde{\delta} & \text{if } k \text{ is odd,} \\
2\tilde{\alpha}_0 + (k-1)\tilde{\delta} & \text{if } k \text{ is even.}
\end{cases}
\]

There are exactly two convex orders on \(\Delta_{im}^+\): the order \(>_{+}\)

\[r_1 >_{+} r_2 >_{+} \cdots >_{+} \delta >_{+} \cdots >_{+} r^2 >_{+} r^1,\]

and the reverse of this order, which we denote by \(>_{-}\).

**Definition 1.24** A rank 2 affine decorated pseudo-Weyl polytope is a pseudo-Weyl polytope along with a choice of two partitions \(a_\delta = (\lambda_1 \geq \lambda_2 \geq \cdots)\) and \(\bar{a}_\delta = (\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \cdots)\) such that \(\mu^\omega - \mu_\infty = |a_\delta|\delta\) and \(\bar{\mu}^\omega - \bar{\mu}_\infty = |\bar{a}_\delta|\delta\). Here \(|a_\delta| = \lambda_1 + \lambda_2 + \cdots\) and \(|\bar{a}_\delta| = \bar{\lambda}_1 + \bar{\lambda}_2 + \cdots\).

**Definition 1.25** The right Lusztig data of a decorated pseudo-Weyl polytope \(P\) is the refinement \(a = (a_\alpha)_{\alpha \in \Delta_+}\) of the Lusztig data from Section 1.3 with respect to \(>_{+}\) (which records the lengths of the edges parallel to each root up one side of \(P\)), where, for
\( \alpha \neq \delta, a_\alpha = a^{\alpha^{-}}_\alpha(P), \) and \( a_\delta \) is as in Definition 1.24. Similarly the left Lusztig data is \( \bar{a} = (\bar{a}_\alpha)_{\alpha \in \tilde{\Delta}}, \) where, for \( \alpha \neq \delta, \bar{a}_\alpha = a^{\alpha^{-}}_\alpha(P), \) and \( \bar{a}_\delta \) is as in Definition 1.24.

Figure 1. An \( \hat{sl}_2 \) MV polytope. The partitions labeling the vertical edges are indicated by including extra vertices on the vertical edges, such that the edge is cut into the pieces indicated by the partition. Here the Lusztig data is

\[
\begin{align*}
\bar{a}_1 &= 2, \quad a_2 = 1, \quad a_3 = 1, \quad \lambda = (9, 2, 1, 1), \quad a^3 = 1, \quad a^1 = 1, \\
\bar{a}_1 &= 1, \quad \bar{a}_2 = 2, \quad \bar{a}_3 = 1, \quad \bar{a}_4 = 1, \quad \bar{\lambda} = (2, 1, 1), \quad \bar{a}^1 = 1, \quad \bar{a}^2 = 1, \quad \bar{a}^3 = 5,
\end{align*}
\]

and all other \( a_k, \bar{a}_k, \bar{a}^k \) are 0.

In [BDKT], the first author and collaborators combinatorially define a set \( MV \) of decorated pseudo-Weyl polytopes, which they call **rank 2 affine MV polytopes**. We will not need the details of this construction, but will instead use the following result from [MT]. Assume \( \mathfrak{g} \) is of type. Define \( \ell_0 \) and \( \ell_1 \) by \( \delta = \ell_0 a_0 + \ell_1 a_1 \) (so \( \ell_0 = \ell_1 = 1 \) for \( \hat{sl}_2 \), and \( \ell_0 = 2, \ell_1 = 1 \) for \( A_2^{(2)} \)).

**Theorem 1.26** [MT, Theorem 3.10] There is a unique map \( b \to P_b \) from \( B(-\infty) \) to type \( \mathfrak{g} \) decorated pseudo-Weyl polytopes (considered up to translation) such that, for all \( b \in B(-\infty) \), the following hold.
Remark 1.27 This theorem implies that, for any rank-2 affine MV polytope, any biconvex order $>$, and any real root $\alpha$, the crystal theoretic Lusztig data $a_\alpha$ agrees with the Lusztig data $a_\alpha$ for the corresponding MV polytope for all real roots $\alpha$. In fact, it follows from Corollary 3.13 below that this holds for all affine algebras, regardless of the rank.

1.6. Khovanov-Lauda-Rouquier algebras. In this section, we recall the basic facts about the Khovanov-Lauda-Rouquier algebras (sometimes called quiver Hecke algebras) attached to the Lie algebra $\mathfrak{g}$ as defined for Kac-Moody algebras in [KL09, Rou], and extended to the case of Borcherds algebras in [KOP].

This is an algebra $R$ built out of generic string diagrams, i.e. immersed 1-dimensional submanifolds of $\mathbb{R}^2$ whose boundary lies on the lines $y = 0$ and $y = 1$, where each string (i.e each immersed copy of the interval) projects homeomorphically to $[0, 1]$ under the projection to the $y$-axis (so in particular there are no closed loops). These are assumed to be generic in the sense that

- no points lie on 3 or more components
- no components intersect non-transversely.

Each string is labeled with a simple root of the corresponding Kac-Moody algebra, and each string is allowed to carry dots at any point where it does not intersect another (but with only finitely many dots in each diagram). All diagrams are considered up to isotopy preserving all these conditions.

Define a product on the space of $\mathbb{k}$-linear combinations of these diagrams, where the product $ab$ of two diagrams is formed by stacking $a$ on top of $b$, shrinking vertically by a factor of 2, and smoothing kinks; if the labels of the line $y = 0$ for $a$ and $y = 1$ for $b$ cannot be isotoped to match, the product is 0.
This product gives the space of $\mathbb{k}$-linear combinations of these diagrams the structure of an algebra, which has the following generators, which depend on a sequence $i = (i_1, \ldots, i_n)$ of nodes of the Dynkin diagram:

- The idempotent $e_i$ which is straight lines labeled with $(i_1, \ldots, i_n)$.
- The element $y^i_k$ which is just straight lines with a dot on the $k$th strand.
- The element $\psi^i_k$ which is a crossing of the $i$ and $i + 1$st strand.

\[
\begin{array}{c|c|c} 
& \cdots & \\
\hline 
i_1 & i_2 & i_n \\
\hline & \cdots \bullet \cdots & \\
\hline 
i_1 & i_j & i_n \\
\hline & \cdots \times \cdots & \\
\hline 
i_1 & i_j & i_{j+1} & i_n \\
\end{array}
\]

In order to arrive at the KLR algebra $R$, we must impose the relations shown in Figure 2. All of these relations are local in nature, that is, if we recognize a small piece of a diagram which looks like the LHS of a relation, we can replace it with the RHS, leaving the rest unchanged. The relations depend on a choice of a polynomial $Q_{ij}(u, v) \in \mathbb{k}[u, v]$ for each pair $i \neq j$. Let $C = (c_{ij})$ be the Cartan matrix of $g$ and $d_i$ be coprime integers so that $d_j c_{ij} = d_i c_{ji}$. We assume each polynomial is homogeneous of degree $\langle \alpha_i, \alpha_j \rangle = -2d_j c_{ij} = -2d_i c_{ji}$ when $u$ is given degree $2d_i$ and $v$ degree $2d_j$. We will always assume that the leading order of $Q_{ij}$ in $u$ is $-c_{ji}$, and that $Q_{ij}(u, v) = Q_{ji}(v, u)$.

In [LV11, 1.1.4], Lauda and Vazirani define an automorphism $\sigma: R \to R$ which up to sign reflects the diagrams through the vertical axis. We let $M^\sigma$ denote the twist of an $R$-module by this automorphism.

While some other aspects of the representation theory of $R$ are quite sensitive to the choice of $\mathbb{k}$ and $Q_{ij}$ (for example, the dimensions of simple modules), none of the theorems we prove will depend on it; the reader is free to imagine that we have chosen their favorite field and worked with it throughout.

Since the diagrams allowed in $R$ never change the sum of the simple roots labeling the strands, it breaks up as a direct sum of algebras $R \cong \bigoplus_{\nu \in Q^+} R(\nu)$, where $Q^+$ is the positive part of the root lattice, and for $\nu = \sum a_i \alpha_i$, $R(\nu)$ is the part of the algebra with exactly $a_i$ strings colored with each simple root $\alpha_i$. In particular, for any simple $R$-module $L$, there is a unique $\nu$ such that $R(\nu) \cdot L = L$. We call this the weight of $L$. We let $L^\mu$ denote the unique 1-dimensional simple of $R(\alpha_i)$.

For any two positive elements of the root lattice $\mu, \nu$, then is an inclusion $R(\mu) \otimes R(\nu) \hookrightarrow R(\mu + \nu)$ given by horizontal juxtaposition. We use

\[
\begin{align*}
\text{Res}^{\mu+\nu}_{\mu, \nu} \, (-) &= \text{Res}_{R(\mu) \otimes R(\nu)} R(\mu^+ + \nu) \, (-) \\
\text{Ind}^{\mu+\nu}_{\mu, \nu} \, (-) &= R(\mu + \nu) \otimes_{R(\mu) \otimes R(\nu)} (-)
\end{align*}
\]

to denote the functors of restriction and extension of scalars along this map.
Figure 2. The relations of the KLR algebra. These relations are insensitive to labeling of the plane.

**Definition 1.28** Fix representations $L$ of $R(\mu)$ and $L'$ of $R(\nu)$; then we have an induced module

$$L \circ L' := \text{Ind}_{\mu,\nu}^{\mu+\nu} (L \otimes L').$$

See [KL09, §2.6] for a more extensive discussion of this functor.
We will only ever consider modules over $R(\nu)$ which are finite dimensional and on which all the $y_k^l$'s act nilpotently; for simple modules, this is equivalent to Lauda and Vazirani’s condition that their modules are gradable. The following result of Lauda and Vazirani is crucial to us:

**Proposition 1.29** ([LV11, Section 5.1]) The set $\mathcal{KLR}$ of isomorphism classes of gradable simple modules over $R$ carry a crystal structure with operators defined by

\[
\tilde{e}_i L = \text{cosoc}(L \circ \mathcal{L}_i) \quad \tilde{f}_i L = \text{cosoc}(\text{Hom}_{R(\nu-\alpha_i) \otimes R(\alpha_i)}(R(\nu - \alpha_i) \otimes \mathcal{L}_i, \text{Res}^\nu_{\nu-\alpha_i, \alpha_i} L)),
\]

\[
\tilde{e}_i^* L = \text{cosoc}(\mathcal{L}_i \circ L) \quad \tilde{f}_i^* L = \text{cosoc}(\text{Hom}_{R(\alpha_i) \otimes R(\nu-\alpha_i)}(\mathcal{L}_i \otimes R(\nu - \alpha_i), \text{Res}^\nu_{\nu, \nu-\alpha_i} L)),
\]

and this bicrystal is isomorphic to $B(-\infty)$. The map $(-)^* : \mathcal{KLR} \to \mathcal{KLR}$ is intertwined with the Kashiwara involution of $B(-\infty)$.

**Remark 1.30** Our conventions are dual to those of [LV11], since we consider $B(-\infty)$ rather than $B(\infty)$.

**Remark 1.31** In [LV11], the operator $\tilde{f}_i$ was actually defined as a socle, not a cosocle; however, as noted by Khovanov and Lauda [KL09, §3.2], all simple modules over the KLR algebra are self-dual, and $\text{Hom}_{R(\nu-\alpha_i) \otimes R(\alpha_i)}(R(\nu - \alpha_i) \otimes \mathcal{L}_i, \text{Res}^\nu_{\nu-\alpha_i, \alpha_i} -)$ commutes with duality, so its socle and cosocle when applied to a simple module are isomorphic.

It is shown in [KL09, 2.5] that, for all $\nu$,

\[
(4) \quad \left\{ \psi_\sigma \left( \prod_{k=1}^n (y_k^l)^{r_k} \right) e_1 \mid \text{wt}(i) = \nu, r_1, \ldots, r_n \geq 0, \sigma \in S_n \right\}
\]

is a basis for $R(\nu)$, where $\psi_\sigma$ is an arbitrary diagram which permutes its strands as the permutation $\sigma$ with no double crossings.

**Remark 1.32** One can consider the “character”

\[
\text{ch}(M) = \sum_i \dim_q(e_i M) \cdot w[i]
\]

as an element of $\mathcal{T}$, the free $\mathbb{C}[q, q^{-1}]$-module generated by words in the nodes of the Dynkin diagram. As shown in [KL09, 2.20], it follows from (4) that

\[
\text{ch}(M_1 \circ M_2) = \text{ch}(M_1)\text{ch}(M_2)
\]

where the product on the right is the usual shuffle product.

2. KLR algebras and Lusztig data

Kleshchev and Ram’s work [KR] studying simple representations of KLR algebras in terms of Lyndon word combinatorics allows one to construct a Lusztig datum for each KLR module with respect to any convex order which arises from a lexicographic order on Lyndon words. We now extend this to obtain a Lusztig datum for any convex...
order. In general, we can no longer use the same type of combinatorics on words that they develop, and instead our main tool is the notion of a cuspidal representation with respect to a “charge”.

2.1. Cuspidal decompositions. Let $i = i_1 \cdots i_n$ be a word in the nodes of the Dynkin diagram and let $\alpha_i = \sum_{k=1}^n \alpha_{i_k}$. Fix a charge $c$, and consider the preorder $<$ on positive elements of the root lattice induced by taking arguments with respect to this charge, as in Section 1.2.

**Definition 2.1** The top of a word $i$ is the maximal element which appears as the sum of a proper left prefix of the word; that is

$$\top (i) = \max_{1 \leq j \leq n} \alpha_{i_1} \cdots \alpha_{i_j}.$$

We call a word in the simple roots $c$-cuspidal if $\top (i) < \alpha_1$ and $c$-semi-cuspidal if $\top (i) \leq \alpha_1$.

Geometrically, we can visualize our word as a path in the weight lattice, and then picture its image in the complex plane under $c$. A word is $c$-cuspidal if this path stays strictly clockwise of the line from the beginning to the end of the word and $c$-semi-cuspidal if stays weakly clockwise of this line, as shown in Figure 3.

**Definition 2.2** The top of a module over $R$ is the maximum among the tops of all $i$ such that $e_i M \neq 0$. We call a simple module over $R(\nu)$ cuspidal if $\top (L) < \nu$, and semi-cuspidal if $\top (L) \leq \nu$.

Obviously, a simple representation is (semi-)cuspidal if and only if all words which appear in its character are (semi-)cuspidal.

**Theorem 2.3** Fix a charge $c$. If $L_1, \ldots, L_h$ are semi-cuspidal representations with $\wt (L_1) > \cdots > \wt (L_h)$, then $L_1 \circ \cdots \circ L_h$ has a unique simple quotient. Furthermore,
every gradable simple appears this way for a unique sequence of semi-cuspidal representations.

**Remark 2.4** Theorem 2.3 holds even if $c$ is not generic.

In order to prove Theorem 2.3, we introduce a more general compatibility condition on representations:

**Definition 2.5** We call an $h$-tuple $(L_1, \ldots, L_h)$ unmixing if

$$\text{Res}_{\nu_1, \ldots, \nu_h}^{\nu_1, \ldots, \nu_h}(L_1 \circ \cdots \circ L_h) = L_1 \boxtimes \cdots \boxtimes L_h.$$

**Lemma 2.6** If $(L_1, \ldots, L_h)$ is an unmixing $h$-tuple, then $L_1 \circ \cdots \circ L_h$ has a unique simple quotient, which throughout we’ll denote $A(L_1, \ldots, L_h)$.

**Proof.** Let $e$ denote the idempotent in $R(\nu)$ projecting to $\text{Res}_{\nu_1, \ldots, \nu_h}^{\nu_1, \ldots, \nu_h}(\nu)$. Then $L_1 \circ \cdots \circ L_h$ is generated by any non-zero vector in the image of $e$; thus, a submodule $M \subset L_1 \circ \cdots \circ L_h$ is proper if and only if it is killed by $e$. It follows that the sum of any two proper submodules is still killed by $e$, and thus again proper. There is thus a unique maximal proper submodule of $L_1 \circ \cdots \circ L_h$, which is to say this module has a simple cosocle. This establishes the theorem. □

**Lemma 2.7** If $L_1, \ldots, L_h$ are semi-cuspidal representations with $\text{wt}(L_1) > \cdots > \text{wt}(L_h)$, then the $h$-tuple $(L_1, \ldots, L_h)$ is unmixing.

**Proof.** Let $i = i_1 i_2 \cdots i_h$ be a word such that $e_i(L_1 \circ \cdots \circ L_h) \neq 0$ and $\text{wt}(i_1) = \nu_1, \ldots, \text{wt}(i_h) = \nu_h$. Fix words $j_1, \ldots, j_h$ such that $e_{i_k} L_k \neq 0$, and so that $i$ is a shuffle of these words. For $1 \leq k, g \leq h$, let $i^g_k$ be the subword of $j^g$ which appear as letters in $i_k$, and let $\nu^g_k = \text{wt}(j^g_k)$.

For each $g$, $\nu^g_1$ is the sum of the roots in a prefix of a word in the character of $L^g$. Thus by semi-cuspidality, $\nu^g_1 \leq \nu_1$ and, for $g > 1$, we have $\nu^g_1 \leq \nu^g_k < \nu_1$. Clearly $\nu^g_1 + \cdots + \nu^g_h = \nu_1$, so this is only possible if $\nu^g_1 = \nu_1$, and so $j_1 = i_1$. Applying this argument inductively, we see that $j_k = i_k$ for all $k$. This immediately implies that $(L_1, \ldots, L_h)$ is unmixing □

**Proof of Theorem 2.3.** By Lemmata 2.6 and 2.7, the induction $L_1 \circ \cdots \circ L_h$ has a unique simple quotient. It remains to show that every simple appears in this way for a unique sequence of semi-cuspidals.

Consider the maximum argument

$$c_{\text{max}} := \max_{e_i L \neq 0} \text{arg}(c(\text{top}(i)))$$

of the top of any word in the character of $L$; let $\nu_1$ be the element of the root lattice greatest height such that $c_{\text{max}}$ is achieved by a prefix of weight $\nu_1$. We’ll prove the
result by induction on the height of $\nu - \nu_1$. If $\nu = \nu_1$, then $L$ is semi-cuspidal, and we are done.

By assumption, $\text{Res}^\nu_{\nu_1,\nu - \nu_1} L \neq 0$, and so this has a simple submodule $L' \otimes L''$. Furthermore, $L'$ is semi-cuspidal (by the definition of $\nu_1$) and $(L', L'')$ is an unmixing pair (by arguments as in the proof of Lemma 2.7). In particular, $L$ is the unique simple quotient of $L' \circ L''$ and the quotient map induces an isomorphism $\text{Res}^\nu_{\nu_1,\nu - \nu_1} L \cong L' \otimes L''$.

By the inductive assumption, $L''$ has a unique semi-cuspidal decomposition $L'' = A(L'_1, \ldots, L'_h)$; thus we have that $L = A(L', L_1, \ldots, L_h)$, so this proves that every simple is of the desired form.

On the other hand, if $L = A(L'_1, \ldots, L'_p)$ for some other cuspidal simples with $\text{wt}(L'_1) > \cdots > \text{wt}(L'_p)$, we cannot have $\text{wt}(L'_1) > \nu_1$, by the maximality of the argument of $\nu_1$. Then we must also have words in the characters of $L'_1$ must have prefixes whose weights add to $\nu_1$; by arguments in the proof of Lemma 2.7, this is only possible if this word is entirely from $L'_1$. By symmetry, there is a word in the character of $L'$ whose weight in $\text{wt}(L'_1)$, so we must have $\text{wt}(L'_1) = \nu_1$.

Thus, $\text{Res}^\nu_{\nu_1,\nu - \nu_1} L \cong L'_1 \otimes A(L'_2, \ldots, L'_p)$; consequently, $L'_1 \cong L'$ and $L'' = A(L'_2, \ldots, L'_p)$. Applying the inductive hypothesis, the uniqueness follows. □

**Definition 2.8** For a fixed charge $c$ and simple $L$, we call the associated simples $(L_1, \ldots, L_h)$ for a fixed charge the $c$-semi-cuspidal decomposition of $L$.

**Corollary 2.9** Fix a charge $c$. The number of $c$-semi-cuspidal representations of weight $\nu$ is the sum

$$\sum_{\nu = \beta_1 + \cdots + \beta_n} \prod_{i=1}^{n} m_{\beta_i}$$

of the product of the root multiplicities over the distinct ways $\nu = \beta_1 + \cdots + \beta_n$ of writing $\nu$ as a sum of positive roots $\beta$, which all satisfy $\arg c(\beta_i) = \arg c(\nu)$.

*Proof.* We proceed by induction on $\rho^\nu(\nu)$. If $\nu$ is a simple root, then the statement is obvious, providing the base case.

In general, the dimension of $\mathcal{U}(n)$ is the number of ways of writing $\nu$ as a sum of multiples of positive roots, so this is the number of simple representations of $R(\nu)$. By the inductive assumption and Theorem 2.3, the number of simple representations of $R(\nu)$ that have a semi-cuspidal decomposition with at least two parts is the number of ways of writing $\nu$ as a sum of multiples of roots $\alpha$ where the arguments $\arg c(\alpha)$ are not all equal. Thus the number of semi-cuspidal simple representations of $R(\nu)$ is the number of ways of writing $\nu$ as a sum of multiples of positive roots all of which have the same argument. □
Corollary 2.10 If $g$ is finite type and $c$ is a generic charge (i.e. a charge such that $\arg c(\alpha) \neq \arg c(\beta)$ for all $\alpha \neq \beta \in \Delta_+$), then there is a unique cuspidal representation $L_\alpha$ of $R(\alpha)$ for each positive root $\alpha$, and no others.

Remark 2.11 The finite-type case of Theorem 2.3 (and thus Corollary 2.10) have been shown independently by McNamara [McN, 3.1]; this has been extended to affine type by Kleshchev in [Kle].

Proof of Corollary 2.10. By Corollary 2.9 the only $\nu$ for which there is a semi-cuspidal representation are $\nu = k\alpha$ for some $k \geq 1$ and $\alpha \in \Delta_+$, and in all these cases there is only one isomorphism class of semi-cuspidal representation. The semi-cuspidal representation $L_\alpha$ of dimension $\alpha$ must in fact be cuspidal, since there is no element of the root lattice on the line from 0 to $\alpha$. □

Remark 2.12 For minimal roots (i.e. roots $\alpha$ such that $x\alpha$ is not a root for any $0 < x < 1$; see section 1.2), the same arguments used in the proof of Corollary 2.10 shows that the root multiplicity coincides with the number of cuspidal representations. However, this is not always the case for example, Section 3.4 gives an example where this is false for $\hat{sl}_2$ with $\nu = 2\delta$.

We also have the following generalized notion of cuspidal representation where we allow any convex order on $\Delta_{+}^{\min}$, not just those coming from charges.

Definition 2.13 Fix a pair $(\alpha, \succ)$ of a minimal positive root and a convex order on $\Delta_{+}^{\min}$. We say a simple representation $L$ of $R(\alpha)$ is $\succ$-(semi)-cuspidal if $L$ is $c$-(semi)-cuspidal for some $(\alpha, \succ)$-compatible charge $c$ (see Definition 1.9).

Proposition 2.14 A simple representation $L$ of $R(\alpha)$ for some positive root $\alpha$ is $\succ$-(semi)-cuspidal if and only if $L$ is $c$-(semi)-cuspidal for all $(\alpha, \succ)$-compatible charges $c$.

Proof. Assume that $L$ is $\succ$-cuspidal, and let $c$ be the $(\alpha, \succ)$ compatible charge from Definition 2.13. Let $c'$ be another $(\alpha, \succ)$-compatible charge, and assume $L$ is not cuspidal for $c'$. Thus, there exists $\beta$ with $\beta \succ c' \alpha$ such that $\text{Res}^\alpha_{\beta, \alpha - \beta} L \neq 0$.

Since $c'$ is $(\alpha, \succ)$-compatible this implies that $\beta > \alpha$. Since $c$ is also $(\alpha, \succ)$ compatible, this implies $\beta > c \alpha$ as well. But $L$ is $c$-cuspidal, so $\text{Res}^\alpha_{\beta, \alpha - \beta} L \neq 0$ is a contradiction. Thus $L$ is in fact cuspidal for $c'$ as well. The same argument carries through for semi-cuspidality. □
Corollary 2.15. For any convex order $>$ on $\Delta^\min_+$, the number of $>$-semi-cuspidal representations of weight $\nu$ is the sum

$$\sum_{\nu = \beta_1 + \cdots + \beta_n} \prod_{i=1}^{n} m_{\beta_i}$$

of the product of the root multiplicities over the distinct ways $\nu = \beta_1 + \cdots + \beta_n$ of writing $\nu$ as a sum of positive roots $\beta_i$ parallel to $\nu$. In particular, if $\mathfrak{g}$ is finite type then there is a unique $>$-cuspidal representation $L_\alpha$ of $R(\alpha)$ for each positive root $\alpha$, and no others.

Proof. This follows immediately from Proposition 2.9 and Corollary 2.10 using some $(\alpha, >)$-compatible charge $c$. $\square$

2.2. Lusztig data in $\mathcal{KLR}$. Fix a convex order $>$. We now strengthen Theorem 2.3 to work for an arbitrary convex order (as opposed to just convex orders coming from charges). This gives a generalization of the notion of Lusztig data commonly used in finite type.

Lemma 2.16. Fix a convex order $>$. Any $h$-tuple $L_1, \ldots, L_h$ of $>$-semi-cuspidal representations with $\text{wt}(L_1) > \cdots > \text{wt}(L_h)$ is unmixing.

Proof. We proceed by induction on $h$. For $k = 1, \ldots, h$, let $\beta_k$ be the minimal root parallel to $\text{wt}(L_k)$. Fix a $(\beta_1, >)$ compatible charge. Let $\nu_i = \text{wt}(L_i)$ and $\nu = \nu_1 + \cdots + \nu_h$.

Assume that

$$\text{Res}_{\nu, \nu_1 - \nu_1} (L_1 \circ \cdots \circ L_h) \neq L_1 \boxtimes (L_2 \circ \cdots \circ L_h).$$

Then there would have to be a word $j$ with $\text{wt}(j)$ a multiple of $\beta_1$, such that $j$ is a shuffle of prefixes of words that appear in the character of $L_k$ as $k$ ranges from 1 to $h$. Since each $L_k$ is $>$-cuspidal and $\beta_1 > \beta_k$ for each $k > 1$, this is impossible. Thus, we must have that

$$\text{Res}_{\nu, \nu_1 - \nu_1} (L_1 \circ \cdots \circ L_h) = L_1 \boxtimes (L_2 \circ \cdots \circ L_h).$$

By induction, we have already assumed that $(L_2, \ldots, L_h)$ is unmixing, so the result follows. $\square$

Theorem 2.17. Fix a convex order $>$. If $L_1, \ldots, L_h$ is an any tuple of $>$-semi-cuspidal representations with $\text{wt}(L_1) > \cdots > \text{wt}(L_h)$, then the induction

$$L_1 \circ \cdots \circ L_h$$

has a unique simple quotient. Furthermore, every gradable simple appears this way for a unique sequence of semi-cuspidal representations.

Proof. By Lemmata 2.6 and 2.16, it is clear that $L_1 \circ \cdots \circ L_h$ has a unique simple quotient. Now we must show that every simple $L$ is of this form of a unique $h$-tuple $L_1, \ldots, L_h$. 22
Consider the root $\alpha \in \Delta_{\text{min}}$ be greatest in the order $>\text{subject to the condition that}$
$\text{Res}_{ma,\nu-ma}^L \neq 0$ for some integer $m$; let $m$ be the maximal integer for which this holds.
Now we induct on the height of $\nu - ma$. If $\nu = ma$, then $L$ is semi-cuspidal, and we are done.

The induction step is exactly as in the proof of Theorem 2.3. We must have that
the restriction $\text{Res}_{ma,\nu-ma}^L \cong L_1 \boxtimes L'$ is an outer tensor product with $L_1$ simple and
semi-cuspidal and $L$ simple. By the inductive hypothesis, the simple $L'$ has a semi-
cuspidal decomposition $L' = A(L_2, \ldots, L_h)$ with $\text{wt}(L_1) > \text{wt}(L_j)$ for all $2 \leq j \leq h$, so $L$
is of the desired form. For any other such decomposition $L = A(L_1', \ldots, L_p')$, symmetry
considerations show that $L_1 \cong L_1'$, and the result follows from applying induction
again. \hfill $\Box$

**Remark 2.18** Theorem 2.17 is a generalization of [KR, Theorem 7.2], which gives
exactly the same sort of description of all simple modules, but only applies to the
convex orders arising from Lyndon words.

### 2.3. Saito reflections on $\mathcal{KLR}$. In this section, we discuss how the Saito reflection
from Section 1.1 works when the underlying set of $B(-\infty)$ is identified with $\mathcal{KLR}$, and
specifically how it interacts with the operation of induction.

Note that if $\varphi_i^*(L_1) = \varphi_i^*(L_2) = 0$, then any composition factor $L$ of $L_1 \circ L_2$ is also
has $\varphi_i^*(L) = 0$ by [KL09, 2.18]; thus, we can consider the action of Saito reflections on
these simples. Now assume that $(L_1, L_2)$ are an unmixing pair (see Definition 2.5), so
$L_1 \circ L_2$ has a unique simple quotient $L$.

**Lemma 2.19** Assume that $\varphi_i^*(L_1) = \varphi_i^*(L_2) = 0$. Then we have that $(\tilde{e}_i^n)L$ is the unique
simple quotient of

$$L^{(n)} = \begin{cases} 
(\tilde{e}_i^n)L_1 \circ L_2 & n \leq \epsilon_i^*(L_1) \\
(\tilde{e}_i^n)_\epsilon(L_1) \circ (\tilde{e}_i^{n-\epsilon_i^*(L_1)})L_2 & \epsilon_i^*(L_1) < n
\end{cases}$$

**Proof.** Since there are no words in the character of $L_1$ or $L_2$ beginning with $i$, the triple
$(\mathcal{L}_i^n, L_1, L_2)$ is unmixing. By Lemma 2.6, the induction $\mathcal{L}_i^n \circ L_1 \circ L_2$ has a unique simple
quotient.

Thus, if we define a surjective map $\mathcal{L}_i \circ L^{(n-1)} \to L^{(n)}$, this will show by induction
that $L^{(n)}$ has unique simple quotient, and that this quotient agrees with $(\tilde{e}_i^n)L$.

If $n \leq \epsilon_i^*(L_1)$, then the map is the obvious one. If $n > \epsilon_i^*(L_1)$, then we use the fact that

$$\mathcal{L}_i \circ (\tilde{e}_i^n)_{\epsilon_i^*(L_1)}L_1 \cong (\tilde{e}_i^n)_{\epsilon_i^*(L_1)}L_1 \circ \mathcal{L}_i,$$

so we have that

$$\mathcal{L}_i \circ L^{(n-1)} \cong (\tilde{e}_i^n)_{\epsilon_i^*(L_1)}L_1 \circ \mathcal{L}_i \circ (\tilde{e}_i^{n-\epsilon_i^*(L_1)})L_2$$

which has an obvious surjective map to $L^{(n)}$. \hfill $\Box$
Lemma 2.20 If \((L_1, L_2)\) is an unmixing pair such that \(\varphi_i^*(L_1) = \varphi_i^*(L_2) = 0\), and \((\sigma_i(L_1), \sigma_i(L_2))\) is also an unmixing pair, then \(\sigma_i(A(L_1, L_2)) = A(\sigma_i(L_1), \sigma_i(L_2))\).

Proof. Let \(L = A(L_1, L_2)\) and \(L' = A(\sigma_i(L_1), \sigma_i(L_2))\); note that these are both simple.

It follows from Proposition 1.4 that, for any element of \(B(\infty)\) with \(\tilde{f}_i^\sigma(M) = 0\),

\[
\varphi_i^*((\tilde{e}_i^\sigma)^n M) + \varphi_i((\tilde{e}_i^\sigma)^n M) - \langle \text{wt}((\tilde{e}_i^\sigma)^n M), \alpha_i^\vee \rangle = \max(0, e_i(M) - n).
\]

Applying Proposition 1.4(iii-v) again gives

\[
\tilde{f}_i^n(\tilde{e}_i^\sigma)^{e_i(M)} M \equiv (\tilde{e}_i^\sigma)^{e_i(M)} \tilde{f}_i^n M \quad \text{and} \quad \tilde{e}_i^n(\tilde{e}_i^\sigma)^{e_i(M)} M \equiv (\tilde{e}_i^\sigma)^{e_i(M)+n} M.
\]

By Lemma 2.19, \((\tilde{e}_i^\sigma)^{e_i(L_1)+e_i(L_2)} L\) is the unique simple quotient of \((\tilde{e}_i^\sigma)^{e_i(L_1)} L_1 \circ (\tilde{e}_i^\sigma)^{e_i(L_2)} L_2\), and \(\tilde{e}_i^{\#(L_1)+\#(L_2)} L'\) is the unique simple quotient of \(\tilde{e}_i^{\#(L_1)} \sigma_i L_1 \circ \tilde{e}_i^{\#(L_2)} \sigma_i L_2\). By the definition of Saito reflection (Definition 1.5),

\[
\tilde{e}_i^{\#(L_1)} \sigma_i L_j \equiv \tilde{e}_i^{\#(L_1)} (\tilde{e}_i^\sigma)^{e_i(L_1)} \tilde{f}_i^{\#(L_1)} L_j \equiv \tilde{e}_i^{\#(L_1)} \tilde{f}_i^{\#(L_1)} (\tilde{e}_i^\sigma)^{e_i(L_1)} L_j \equiv (\tilde{e}_i^\sigma)^{e_i(L_1)} L_j,
\]

where the middle step uses (5). Thus

\[
(\tilde{e}_i^{\#(L_1)+\#(L_2)} L = (\tilde{e}_i^{\#(L_1)+\#(L_2)} L') \quad \text{by (6)}
\]

It follows that

\[
\sigma_i L \equiv \tilde{e}_i^{\#(L_1)} (\tilde{e}_i^\sigma)^{e_i(L_1)} L \quad \text{by (5)}
\]

\[
\equiv \tilde{e}_i^{\#(L_1)+\#(L_2)} \tilde{e}_i^{\#(L_1)+\#(L_2)-\#(L_1)} (\tilde{e}_i^\sigma)^{e_i(L_1)} L \quad \text{by additivity of weights}
\]

\[
\equiv \tilde{e}_i^{\#(L_1)+\#(L_2)} (\tilde{e}_i^\sigma)^{e_i(L_1)+e_i(L_2)} L \quad \text{by (5)}
\]

\[
\equiv \tilde{e}_i^{\#(L_1)+\#(L_2)} \tilde{e}_i^{\#(L_1)+\#(L_2)} L' \quad \text{by (6)}
\]

\[
= L'.
\]

This completes the proof. \(\square\)

Proposition 2.21 Fix a simple module \(L\) with \(\tilde{f}_i L = 0\), and let \((L_1, \ldots, L_h)\) be its semi-cuspidal decomposition with respect to a fixed convex order \(\succ\) with \(\alpha_i\) greatest. Then the semi-cuspidal decomposition of \(\sigma_i(L)\) for \(\succ^h\) is \((\sigma_i L_1, \ldots, \sigma_i L_h)\). In particular, the operation \(\sigma_i\) defines a bijection between semi-cuspidal modules for \(\succ\) with \(\tilde{f}_i L = 0\) and semi-cuspidal modules for \(\succ^h\) with \(\tilde{f}_i L = 0\). The inverse of this bijection is \(\sigma_i^*\).

Proof. The proof is by induction, fixing \(L\), and assuming the proposition for any simple which has a smaller maximal height in its semi-cuspidal decomposition, or a smaller number of components.

If \(h > 1\), let \(L' = A(L_2, \ldots, L_h)\). Then \((L_1, L')\) is an unmixing pair, and \(L = A(L_1, L')\). By induction, the modules \(\sigma_i L_j\) are semi-cuspidal and \(\sigma(L') = A(\sigma_i L_2, \ldots, \sigma_i L_h)\). Thus, \(\sigma_i(L_1)\) and \(\sigma_i(L')\) are again an unmixing pair, and by Lemma 2.20, we have that \(\sigma_i(L) = A(\sigma_i(L_1), \sigma_i(L'))\). Thus, we have that \(\sigma_i(L) = A(\sigma_i L_1, \ldots, \sigma_i L_h)\).
If $L$ is semi-cuspidal, then we need only establish that $\sigma_i L$ is again semi-cuspidal. Since the sets of semi-cuspidals for $>\nu$ of weight $\nu$ and for $>^0\nu$ of weight $s_i\nu$ have the same number, and we know $\sigma$ is a bijection between the set of simples $L'$ with $\tilde{f}_i L' = 0$ and those with $\tilde{f}_i L = 0$, we can instead show if $L'$ has the same weight as $L$ but is not semi-cuspidal, then $\sigma_i L'$ is not semi-cuspidal. But in that case $L'$ has a smaller maximal height in its cuspidal decomposition then $L$, so $\sigma_i L = A(\sigma_i L_1, \ldots, \sigma_i L_h)$ by induction. □

Put another way, we have that:

**Corollary 2.22** Assume $>$ is a convex order such that $\alpha_i$ is greatest and that $L_1, \ldots, L_h$ are $>-\text{semi-cuspidal}$ representations with $\alpha_i > \text{wt}(L_1) > \cdots > \text{wt}(L_h)$. Then

$$\sigma_i A(L_1, \ldots, L_h) \cong A(\sigma_i L_1, \ldots, \sigma_i L_h).$$

**Remark 2.23** As was recently explained by Kato [Kat], in symmetric type there are in fact equivalences of categories

$$(7) \quad \left\{ L : F_i(L) = 0 \right\} \leftrightarrow \left\{ L : F_i^*(L) = 0 \right\}$$

which induce Saito reflections on the set of simples. Kato’s proof uses the geometry of quiver varieties, which is why it is only valid in symmetric type, but it seems likely that there is an algebraic version of Kato’s functor as well, which should extend his result to all symmetrizable types. We feel this should give an alternative and perhaps more satisfying explanation for Proposition 2.21 and Corollary 2.22.

Corollary 2.22 is a very important technical tool for us. In particular it allows us to reduce questions about cuspidal representations to the case where the root is simple, using the following.

**Lemma 2.24** Fix a simple $L$ and a convex order $>$, and assume the semi-cuspidal decomposition of $L$ is $L = A(L_1, \ldots, L_h)$.

Assume $L_1 = \mathcal{L}_1^n$ for some real root $\alpha$. If $g$ is finite type or is affine with $\alpha > \delta$, then there is a finite sequence $\sigma_{i_1}, \ldots, \sigma_{i_k}$ of Saito reflections such that $s_{i_k} \cdots s_{i_1} \alpha$ is a simple root $\alpha_{m'}$, for each $j$ we have $q_{i_j}^* (\sigma_{i_{j-1}} \cdots \sigma_{i_1} L) = 0$, and

$$\sigma_{i_k} \cdots \sigma_{i_1} L = A(\mathcal{L}_m^{n'}, \ldots, \sigma_{i_k} \cdots \sigma_{i_1} L_h).$$

If instead $L_h = \mathcal{L}_p^p$ for some real root $\beta$ and $g$ is finite type or affine with $\beta < \delta$, then a similar list of dual Saito reflections $\sigma_{i_1}^*, \ldots, \sigma_{i_h}^*$ exists with

$$\sigma_{i_h}^* \cdots \sigma_{i_1}^* L = A(\sigma_{i_h}^* \cdots \sigma_{i_1}^* L_1, \ldots, \mathcal{L}_m^p).$$

**Proof.** The two statements are swapped by the Kashiwara involution, so we need only prove the first. We proceed by induction on the number of positive roots $\eta > \alpha$ (which is thus finite). The case when $\alpha$ is greatest with respect to $>$ (and hence is
simple) is trivially true; so assume that for some $k \geq 1$ the statement is known for all charges $c$ and all positive roots $\alpha$ with at most $k - 1$ positive roots $\eta > \alpha$.

Fix $c$ and $\alpha$ with exactly $k$ roots $> \alpha$. Let $\alpha_i$ be the greatest root (which is necessarily simple). Then $\varphi'_{\alpha_i}(L) = 0$, since $\mathcal{L}_{\alpha_i}$ does not appear in its cuspidal decomposition, and so we can apply Corollary 2.22. This reduces to the same questions with induction of semi-cuspidals $
abla$.

**Proof.** Choose a function $\varphi$ that obtains its minimum on $P_L$ exactly on $F$, and consider the charge $\rho \nu + \iota \varphi$; the simple $L$ is the unique simple quotient of an increasing induction of semi-cuspidals $L_1 \circ \cdots \circ L_h$; let $k$ be smallest index where $\varphi(\nu(L_k)) = 0$ and $m$ the largest such index. Let $L'$ be the simple quotient of $L_1 \circ \cdots \circ L_{k-1}$, let $L_f$ be the simple quotient of $L_k \circ \cdots \circ L_m$, and let $L''$ be the simple quotient of $L_{m+1} \circ \cdots \circ L_h$.

3. KLR polytopes and MV polytopes

Having developed this combinatorics for understanding representations of KLR algebras, we now turn to encoding this information in polytopes.

3.1. KLR polytopes.

**Definition 3.1** For each simple $L$, the character polytope $P_L$ is the convex hull of the weights $\nu'$ such that $\text{Res}^\nu_{\nu',\nu'} L \neq 0$.

**Remark 3.2** Recalling the definition of the character $\text{ch}(L)$ of $L$ from Remark 1.32, we can think of every word $i$ appearing in $\text{ch}(L)$ as a path in $\mathfrak{h}^*$; the polytope $P_L$ can also be described as the convex hull of all these paths. This explains our terminology.

The polytopes $P_L$ live in the dual of the Cartan $\mathfrak{h}^*$, which has a natural height function $\rho \nu$. This orients each edge of the polytope, and gives every face $F$ a highest vertex $v_L$ and lowest vertex $v_b$. We associate a KLR algebra $R_F$ to each face $F$ by $R_F := R(v_L - v_b)$. Thus for each face $F$ of $P_L$, we have the subalgebra of $R(v)$ given by $R(v_b) \otimes R_F \otimes R(v - v_l)$, and can consider the restriction functor $\text{Res}^\nu_F$ restricting to this subalgebra.

**Proposition 3.3** For any simple $L$ and face $F$ of $P_L$, the restriction $\text{Res}^\nu_F L$ is simple and thus the outer tensor of three simples $L' \otimes L'' \otimes L'$. 

**Proof.** Choose a function $\phi$ that obtains its minimum on $P_L$ exactly on $F$, and consider the charge $\rho \nu + \iota \varphi$; the simple $L$ is the unique simple quotient of an increasing induction of semi-cuspidals $L_1 \circ \cdots \circ L_h$; let $k$ be smallest index where $\varphi(\nu(L_k)) = 0$ and $m$ the largest such index. Let $L'$ be the simple quotient of $L_1 \circ \cdots \circ L_{k-1}$, let $L_f$ be the simple quotient of $L_k \circ \cdots \circ L_m$, and let $L''$ be the simple quotient of $L_{m+1} \circ \cdots \circ L_h$. 

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So, the module $L' \circ L_F \circ L''$ is a quotient of $L_1 \circ \cdots \circ L_h$, and thus has a unique simple quotient, which is $L$. On the other hand, $\text{Res}_F^L(L' \circ L_F \circ L'') = L' \boxtimes L_F \boxtimes L''$, so $L$ must also restrict to this same module.

**Definition 3.4** Fix $L \in \mathcal{KLR}$. The **KLR polytope** $\hat{P}_L$ of $L$ is the polytope $P_L$ along with the data of the isomorphism class of the semi-cuspidal representation $L_E$ associated to each edge $E$ of $P_L$ as above. We denote by $P^\mathcal{KLR}$ the set of all KLR polytopes.

**Remark 3.5** The representations which can appear as labels in $\hat{P}_L$ are not arbitrary; they must be semi-cuspidal for any charge which includes that edge in its walk.

**Proposition 3.6** Every edge of $P_L$ is parallel to a positive root of $\mathfrak{g}$. That is, $P_L$ is a pseudo-Weyl polytope.

**Proof.** For any edge $E$, we can pick a generic function $\varphi$ which achieves its maximum on $P_L$ exactly on $E$. Since at most one element of $\Delta_{\text{min}}^+$ is parallel to $E$, a generic such $\varphi$ produces a charge generic in the usual sense: it induces a total order on the words appearing in simple representations of $R_E$. Furthermore, $L_E$ is semi-cuspidal for this charge so by Corollary 2.9 is a multiple of a positive root.

**Remark 3.7** In finite type, there is exactly one semi-cuspidal simple representation of $R(k\alpha)$ for each positive root $\alpha$ and $k \geq 1$, so the decoration is superfluous. The KLR polytope $\hat{P}_L$ is completely determined by the character polytope $P_L$, and $P^\mathcal{KLR}$ can be thought of as simply a set of pseudo-Weyl polytopes.

Recall that that, as in Lemma 1.15, each convex order $\succ$ in Lemma 1.15 defines a path $P_L^\succ$ through $P_L$. We obtain a list simple modules of simple modules $L_1, \ldots, L_h$ with $\text{wt}(L_1) \succ \cdots \succ \text{wt}(L_h)$ by taking the modules corresponding to the edges in $P^\succ$.

**Proposition 3.8** For any simple $L$, we have $L = A(L_1, \ldots, L_h)$, where $L_1, \ldots, L_h$ are as described above.

**Proof.** We induct on $h$, the case when $L$ is $\succ$-semi-cuspidal being obvious. Let $E$ be the last edge in $P^\succ$, and consider $\text{Res}_E^L L$; this is of the form $L' \boxtimes L_h$. Obviously the edges in $P_L$ and $\hat{P}_L$ coincide along the walk corresponding to $\succ$ up to but not including $E$. Thus, $L_1, \ldots, L_{h-1}$ are the simples associated to this walk for $L'$ by the algorithm above, and by the inductive assumption, $L' = A(L_1, \ldots, L_{h-1})$. Thus, $A(L_1, \ldots, L_h)$ is the unique simple quotient of $L' \circ L_h$. Of course, $L$ is also a quotient of this module by Frobenius reciprocity, so these simples coincide.

**Proposition 3.8** has the following immediate consequences:
Corollary 3.9 For any $\succ$, the polytope $P_L$ with the labeling of just its edges along $P^\succ$ uniquely determines the simple $L$. In particular, the map $L \mapsto \tilde{P}_L$ defines a bijection $\mathcal{KLR} \to P^{\mathcal{KLR}}$. □

Corollary 3.10 The function sending a labelled polytope to the list of semi-cuspidal representations attached to $P^\succ$ is a bijection from $P^{\mathcal{KLR}}$ to the set of ordered lists of semi-cuspidal representations, for any convex order. □

Since the map which takes $L$ to $\tilde{P}_L$ is injective, the crystal structure on $\mathcal{KLR}$ gives rise to a crystal structure on $P^{\mathcal{KLR}}$. Using Corollary 3.9, we can now give a combinatorial description of the resulting crystal operators.

Proposition 3.11 To apply the operator $\tilde{f}_i$ to $\tilde{P} \in P^{\mathcal{KLR}}$, we choose a convex order with $\alpha_i$ lowest, and read the path determined by that order to obtain a list of semi-cuspidal representations $L_1, \cdots, L_h$ corresponding to increasing roots in that order. If $L_h = \mathcal{L}_i^k$ for some $k \geq 1$, then

$$\tilde{f}_i \tilde{P} = \tilde{P}_{A(L_1, \cdots, L_{h-1}, \mathcal{L}_i^{k-1})}.$$ 

If $L_h \neq \mathcal{L}_i^k$, then $\tilde{f}_i \tilde{P} = 0$.

Proof. If $L_h \neq \mathcal{L}_i^k$, then $A(L_1, \ldots, L_h)$ is a quotient of $L_1 \circ \cdots \circ L_{h,i}$, whose character is a quantum shuffle of words not ending in $i$, and thus only contains words not ending in $i$. Thus, $\tilde{f}_i A(L_1, \ldots, L_h) = 0$ follows immediately. On the other hand $A(L_1, \ldots, L_{h-1}, \mathcal{L}_i^k)$ is a quotient of

$$A(L_1, \ldots, L_{h-1}) \circ \mathcal{L}_i^k \cong (A(L_1, \ldots, L_{h-1}) \circ \mathcal{L}_i^{k-1}) \circ L_i \to A(L_1, \ldots, L_{h-1}, \mathcal{L}_i^{k-1}) \circ \mathcal{L}_i,$$

and thus by definition is $\tilde{e}_i A(L_1, \ldots, L_{h-1}, \mathcal{L}_i^{k-1}) = A(L_1, \ldots, L_h)$ and we are done. □

We also have the following, which is simply a restatement of Corollary 2.22 in the language of polytopes.

Proposition 3.12 To apply a Saito reflection functor $\sigma_i$ to a polytope $\tilde{P} \in P^{\mathcal{KLR}}$ with $\tilde{f}_i \tilde{P} = 0$, choose a convex order with $\alpha_i$ greatest and let $L_1, \cdots, L_h$ be as before. Then

$$\sigma_i \tilde{P} = \tilde{P}_{A(\sigma_i L_1, \cdots, \sigma_i L_h)}.$$ 

□

Comparing Propositions 3.11 and 3.12 with the definition of crystal theoretic Lusztig data 1.11, it is immediate that:

Corollary 3.13 For any simple $L$ with corresponding element $b$ in $B(-\infty)$ and any convex order $\succ$, the geometric Lusztig data $a_d(P_L)$ from Definition 1.16 agrees with the crystal-theoretic Lusztig data $a_{\alpha}(b)$ from Definition 1.11 for all real roots which
are greater than any imaginary root, and with the dual data $a_\alpha^*(b)$ for all real roots which are less than any imaginary root.

**Proof of Theorem A.** Two pseudo-Weyl polytopes for a finite dimensional root system coincide if and only if their Lusztig data are identical for every convex order. By Corollary 1.20 the Lusztig data of the MV polytope corresponding to $b$ is given by the crystal-theoretic Lusztig data $a_\alpha(b)$, and that of the KLR polytope $P_L$ is given by $a_\bullet(L)$ by definition. Thus Corollary 3.13 shows that these polytopes coincide.

We can generalize this theorem a little bit; fix a charge $c$ such that every imaginary root $\alpha$ has $\arg(c(\alpha)) < \pi/2$; we can, of course, symmetrically deal with the faces where $\arg(c(\alpha)) > \pi/2$ instead. While this might seem like a strange and restrictive condition, it has a natural interpretation. These are precisely the faces parallel to a face of the weight polytope of a lowest weight representation (or highest weight, for the reversed condition). In affine type, we can split faces into two groups: either they have this form, or they are parallel to $\delta$. In hyperbolic type, the situation will be considerably more complicated.

**Proposition 3.14** For each $L \in \text{KLR}$, the face $F$ of $P_L$ defined by $c$ is an MV polytope for $g_c$.

**Proof.** Of course, we can assume that $L$ is semi-cuspidal with argument $\pi/2$ without loss of generality. By Lemma 2.24, we can find a list of reflections $s_{i_1}, \ldots, s_{i_k}$ such that for $c^{i_1 \cdots i_k}$, the roots $s_{i_1} \cdots s_{i_k} \beta_j$ are all greater than all other roots in the corresponding convex preorder. Thus, these roots must all be simple, and we have that $s_{i_1} \cdots s_{i_k} F$ is a face of the polytope $P_{s_{i_1} \cdots s_{i_k} L}$ by Proposition 3.12. On the other hand, $s_{i_1} \cdots s_{i_k} L$ is a representation on a KLR algebra for the finite type algebra $g_{c^{i_1 \cdots i_k}}$; we simply don’t use any strands labeled with other roots. Thus, $P_L$ is an MV polytope for $g_{c^{i_1 \cdots i_k}}$ algebra by Theorem A, and thus so is $F$. 

3.2. *The crystal corresponding to a face.* Fix a charge $c$. The set of real roots with argument $\pi/2$ are the real roots of some root system $\Delta_c$. Let $g_c$ be the corresponding Lie algebra. Let $\beta_1, \ldots, \beta_\Delta$ be the simple roots of $g_c$. To avoid the possibility of confusion between the indexing set of the roots of $g$ and those of $g_c$, we will always index the latter with underlined numbers.

**Remark 3.15** As we discuss in the Section 3.6, in general it may be better to associate to a face all roots of argument $\pi/2$, which in general is the root system of a Borcherds algebra of possibly infinite rank. However, our main goal here is to understand affine type, and there defining $g_c$ as we do is more convenient.

**Definition 3.16** Fix a charge $c$. The face crystal $\text{KLR}[c]$ for $c$ is the set of $c$-semi-cuspidal representations $L$ of argument $\pi/2$. 

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Notice that \( \mathcal{KLR}[c] \) consists exactly of those representations which occur as the representation \( L_F \) associated in the previous section to the face \( F \) of \( P_L \) defined by the charge \( c \) and argument \( \pi/2 \), where \( L \) is some simple. This justifies the term “face” in Definition 3.16. The term “crystal” is justified by the following:

**Proposition 3.17** The operators given by
\[
\tilde{e}_i L = \cosoc(L \circ L_{\beta_i}), \quad \tilde{\varphi}_i L = \cosoc(L_{\beta_i} \circ L), \\
\tilde{f}_i L = \soc(Hom_{R^{(v)}}(R \circ L_{\beta_i} L)), \quad \tilde{f}^*_i L = \soc(Hom_{R^{(v)}}(L_{\beta_i} \circ R, L))
\]
define a \( g_+ \) combinatorial bicrystal structure on \( \mathcal{KLR}[c] \). The additional combinatorial data is given by
\[
\underline{\text{wt}}(L) = \text{wt}(L)_{|h_0},
\]
\[
\varphi^*_i(L) = \max\{n \mid \tilde{f}^*_i(L)^n(\varphi_i L) \neq 0\}, \quad \varphi_i(L) = \max\{n \mid (\tilde{f}^i)^n(L) \neq 0\},
\]
\[
e_i(L) = \varphi_i(L) - \alpha^\vee_i(\tilde{\text{wt}}(L)), \quad e_i(L) = \varphi^*_i(L) - \alpha^\vee_i(\text{wt}(L)).
\]

**Proof.** Our principal task is to show that these operators are well defined.

Fix \( L \) and let \( v = \text{wt}(L) \). For each simple root \( \beta_i \) of \( \Delta_c \), we can choose a deformations \( c_\pm \) of the charge \( c \) such that:

(i) for some small \( \varepsilon > 0 \), elements \( \mu \) of the weight lattice with \( v - \mu \in \text{span}_{\mathbb{Z}_{\geq 0}}(\alpha_i) \) have
- \( \arg(c_+(\mu)) \in (\pi/2 - \varepsilon, \pi/2 + \varepsilon) \) if and only if \( \arg(c(\mu)) = \pi/2 \),
- \( \arg(c_-(\mu)) > \pi/2 + \varepsilon \) if and only if \( \arg(c(\mu)) > \pi/2 \),
- \( \arg(c_-(\mu)) < \pi/2 - \varepsilon \) if and only if \( \arg(c(\mu)) < \pi/2 \).

(ii) the root \( \beta_i \) is greater for \( >_{c_+} \) and lesser for \( >_{c_-} \) than all other roots \( \beta \neq \beta_i \) with \( \arg(c(\beta)) = \pi/2 \).

Note that we cannot require these conditions for all \( \mu \) if \( g \) is not finite type. For example, if \( g \) is affine and \( \arg(c(\delta)) = \pi/2 \), for any real root with \( \arg(c(\beta)) \neq \pi/2 \) we must have \( \arg(c_\pm(\beta + k\delta)) \in (\pi/2 - \varepsilon, \pi/2 + \varepsilon) \) for \( k \gg 0 \). Of course, \( v - \beta - k\delta \notin \text{span}_{\mathbb{Z}_{\geq 0}}(\alpha_i) \) for \( k \gg 0 \), so this is not a fatal problem.

For each \( L \in \mathcal{KLR}[c] \), we have semi-cuspidal decompositions for \( c_\pm \). The conditions on \( c_\pm \) imply that every representation which appears in these must be itself in \( \mathcal{KLR}[c] \).

Thus, we have a surjective map \( \cdots \circ L_{\beta_i}^{k+1} \rightarrow L \circ L_{\beta_i} \) and similarly for \( L_{\beta_i} \circ L \). By Theorem 2.3, this shows that \( L \circ L_{\beta_i} \) and \( L_{\beta_i} \circ L \) have unique simple quotients; in fact, we can define our crystal operators by
\[
\tilde{e}_i L = A(L_{\beta_i}^{n+1}, \ldots) \quad \tilde{\varphi}_i L = A(\ldots, L_{\beta_i}^{k+1}).
\]

Condition (i) of Definition 1.1 is tautological. Condition (ii) follows from the observation that \( k = \varphi_i(L) \) and \( n = \varphi^*_i(L) \), so indeed these increase as expected with \( \tilde{e}_i \) and \( \tilde{\varphi}_i \).
Now, consider $\text{Res}^{\nu+\beta_i}_{\nu_i} \tilde{\theta}_i L$; the socle of this module is of the form $L' \boxtimes \mathcal{L}_{\beta_i}$ for some semi-simple module $L'$, and

$$L' = \text{soc}(\text{Hom}_{R(\nu+\beta_i)}(R \circ \mathcal{L}_{\beta_i}, \tilde{\theta}_i L)).$$

First of all, $L'$ can have no simple summands other than $L$, since if $L''$ where such a simple, we would have a surjection $L'' \circ \mathcal{L}_{\beta_i} \to L$, which is impossible by Theorem 2.3. On the other hand, $\text{Hom}(L, L') \cong \text{Hom}(L \circ \mathcal{L}_{\beta_i}, \tilde{\theta}_i L) \cong k$, so $L$ has multiplicity 1 in $L'$, and so $L \cong L'$; thus condition (iii) holds for unstarrred operators, and exactly the same argument shows it for starred operators. Finally, condition (iv) is vacuous in this case. □

The remainder of Section 3.2 is devoted to proving results on the structure of the face-crystal in special cases. These cases will be crucial to understanding the finite and affine type situation later on.

**Lemma 3.18** Fix a charge $c$ and a simple root $\alpha_i$. If $\alpha_i$ is the greatest root for $c$ then Saito reflection $\sigma_i$ induces a bicrystal isomorphism between $\mathcal{KLR}[c]$ and $\mathcal{KLR}[c^s]$. Similarly, if $\alpha_i$ is the lowest root for $c$, then $\sigma_i^*$ induces a bi-crystal isomorphism between $\mathcal{KLR}[c]$ and $\mathcal{KLR}[c^s]$.

**Proof.** Of course, we need only check this for one set of operators $\tilde{\theta}_i$ and $\tilde{\theta}_i^*$ at a time. Thus, fix $\beta_i$ as before.

We can choose $c_\pm$ as in the proof of Proposition 3.17; we also require that $\alpha_i$ is still maximal for both these orders. For any simple module $L \in \mathcal{KLR}[c]$, we have semi-cuspidal decompositions for $c_\pm$ given by

$$L = A(\mathcal{L}_{\beta_i}^n, \ldots) = A(\ldots, \mathcal{L}_{\beta_i}^k).$$

We have that

$$\tilde{\theta}_i L = A(\mathcal{L}_{\beta_i}^{n+1}, \ldots) \quad \tilde{\theta}_i^* L = A(\ldots, \mathcal{L}_{\beta_i}^{k+1}).$$

It follows immediately from Lemma 2.24 that these operations commute with Saito reflection. □

We say that an element of a bicrystal is **lowest weight** if it is killed by all lowering Kashiwara operators, both starred and unstarrred.

**Lemma 3.19** Assume $L^h$ is a $c$-semi-cuspidal representation of argument $\pi/2$ which is lowest weight for the bicrystal structure with $\text{wt}(L^h)$ a null vector. Then the component generated by $L^h$ under the crystal operators $\tilde{\theta}_i$ is the same as the component generated by $L^h$ under the $\tilde{\theta}_i^*$. 31
Proof. The proof is by induction on the sum $d(L)$ of the coefficients of the expression for $\text{wt}(L) - \text{wt}(L^h)$ in terms of the $\beta_k$. The proof is symmetric in the two structures, so it suffices to fix $L = \hat{e}_j \hat{e}_{j-1} \cdots \hat{e}_1 L^h$ and show that it is in the starred component of $L^h$.

If $d(L) = 1$ then $\text{wt}(L) - \text{wt}(L^h) = \beta_j'$ and $L = \tilde{f}_j L^h$ for some $j$. As in the proof of Proposition 3.17 we can apply Saito reflections until $\beta_k = \alpha_i$. In that case it follows from Proposition 1.4 and the fact that the whole crystal is $B^0(-\infty)$ that $\tilde{f}_j L^h = \tilde{f}_j^* L^h$, so the claim holds.

Now assume the result holds for all $L'$ with depth $d(L') < d$. By the $d = 1$ case this is isomorphic to $L = \hat{e}_{j_1} \hat{e}_{j_{-1}} \cdots \hat{e}_1 L^h$. From here it is clear that $\text{Res}_{\beta_{j_1}, \text{wt}(L)}^\sim L \neq 0$; any simple submodule of this restriction is of the form $\mathcal{L}_{\beta_{j_1}} \otimes L'$ for some $L'$. Thus, we have that $\hat{e}_1 L' = L$, so $\tilde{f}_j L \neq 0$. Since we know that $\mathcal{KLR}$ is in isomorphic to $B^0(-\infty)$ as a bi-crystal, by Proposition 1.4 we must be in one of the following two cases:

1. If $j_1 \neq j_d$ or $j_1 = j_d$ and $\tilde{f}_j, \tilde{f}_d L = \tilde{f}_j, \tilde{f}_d L$ then $L = \hat{e}_1 \hat{e}_{j_1} \hat{f}_j \hat{e}_d L$. The module $\tilde{f}_j L$ is manifestly in the component of the unstarred component of $L^h$, and thus by induction in the starred component as well. Using the inductive hypothesis again, $\hat{e}_1 \hat{f}_j L$ is in the starred components of $L^h$, and so $L$ is as well.

2. If $j_1 = j_d$ and $\tilde{f}_j, \tilde{f}_d L = \tilde{f}_j, \tilde{f}_d L$, then $\tilde{f}_j, \tilde{f}_d L$ is in both the starred and unstarred component of $L^h$. Since $L = \hat{e}_1 \hat{f}_j L$, we see that $L$ is also in the starred component. $\square$

Proposition 3.20 Assume $L^h \in \mathcal{KLR}[c]$ is lowest weight for the bicrystal structure, and that $\text{wt}(L^h)$ is a null vector. Then the component generated by $L^h$ under all $\hat{e}_1 \hat{e}_d$ is isomorphic (as a bicrystal) to the infinity crystal $B^\infty(-\infty)$.

Proof. By Lemma 3.19, it suffices to check conditions (ii)-(vi) of Corollary 1.4. To check condition (ii), consider the module $\mathcal{L}_{\beta_1} \circ L \circ \mathcal{L}_{\beta_1}$; both $\hat{e}_1 \hat{e}_d L$ and $\hat{e}_d \hat{e}_1 L$ are quotients of this module. Since $\beta_1$ and $\beta_2$ are simple among the roots with $c$-argument $\pi/2$, there is a deformation $c'$ of $c$ such that $\beta_2$ is lowest among the roots with $\text{arg}(c(\beta)) = \pi/2$, $\beta_2$ is greatest. Let $(L_1, \ldots, L_n)$ be the semi-cuspidal decomposition of $L$ with respect to $c'$. By assumption, each $L_i$ is semi-cuspidal for $c$, so their weights have argument between that of $c'(\beta_i)$ and $c'(\beta_j)$ (inclusive). Thus, $\mathcal{L}_{\beta_1} \circ L_1 \circ \cdots \circ L_n \circ \mathcal{L}_{\beta_1}$ has a unique simple quotient by Theorem 2.3. Since $\mathcal{L}_{\beta_1} \circ L \circ \mathcal{L}_{\beta_1}$ is a quotient of this module, this uniqueness proves $\hat{e}_d \hat{e}_1 L = \hat{e}_e \hat{e}_1 L$.

Each of the conditions (iii)-(vi) only involves a single simple root $\beta_i$. Furthermore, each of these conditions holds before Saito reflection if and only it holds after by Corollary 2.22. Thus, as in the proof that operators are well-defined (Proposition 3.17), we can apply Saito reflections until $\beta_i$ is simple, in which case these conditions follow from the isomorphism of $\mathcal{KLR}$ with $B^0(-\infty)$ for $g$. $\square$
Corollary 3.21  Fix a charge $c$ such that every root with argument $\pi/2$ is real (so in particular $\mathfrak{g}_c$ is of finite type). Then $\mathcal{KLR}[c]$ with the operators from Proposition 3.17 is isomorphic as a bicrystal to $B^k(-\infty)$.

Proof. The trivial representation satisfies the conditions of Proposition 3.20, so generates a copy of $B^k(-\infty)$. Since all roots of $\mathfrak{g}_c$ are real, the number of these representations of a given weight is exactly the Kostant partition function of $\mathfrak{g}_c$, so this exhausts $\mathcal{KLR}[c]$. □

Corollary 3.22  If both $\mathfrak{g}$ and $\mathfrak{g}_c$ are of affine type, then $\mathcal{KLR}[c]$ is isomorphic as a bicrystal to a direct sum of copies of $B^k(-\infty)$, all lowest weight elements $L^h$ have wt$(L^h) = k\delta$ for some $k$, and the number of lowest weight elements of weight $k\delta$ is the number of $q$-multipartitions of $k$, where $q = r - s = rk_\mathfrak{g} - rk_{\mathfrak{g}_c}$.

Proof. By Proposition 3.20 it suffices to show that all lowest weight elements for the unstarred $\mathfrak{g}_c$ crystal structure are also lowest weight for the starred $\mathfrak{g}_c$ crystal structure, that they must have weight $k\delta$, and that there are the right number for each $k$. We proceed by induction on weight of (potential) lowest weight elements. In weight 0, there is one lowest weight element (the trivial representation), and it satisfies all the conditions. So, assume these conditions hold for all lowest weight elements of depth $\leq k\delta$ for some $k \geq 0$. By Proposition 3.20, each of these lowest weight elements generate a copy of $B^k(-\infty)$. The generating function for the number of $c$-stable reps of argument $\pi/2$ is exactly

$$a(t) = \prod_{\alpha \in \Delta_c} \frac{1}{(1 - t^\alpha)^{\dim_{\mathfrak{g}_c}}}. $$

By comparing with the Kostant partition function

$$b(t) = \prod_{\alpha \in \Delta_c} \frac{1}{(1 - t^\alpha)^{\dim(\mathfrak{g}_c)_\alpha}}$$

for $\mathfrak{g}_c$, we see that

$$\frac{b(t)}{a(t)} = \prod_{k \geq 1} \frac{1}{(1 - t^{k\delta})^q} $$

is just the generating function of the number of $q$-multipartitions with variable $t^\delta$. We know that at $k\delta$ and below, the number of lowest weight elements of the bicrystal structure exactly matches this partition function. Thus, we see that the copies of $B^k(-\infty)$ exhaust all $c$-stable elements of depth less than $(k + 1)\delta$, and miss exactly the number of $q$-multipartitions of $k + 1$ in that depth. These must all be lowest weight for both structures, and satisfy the necessary conditions, so the induction proceeds. □

Proposition 3.23  Assume $\mathfrak{g}$ is of affine type. Fix $M, N \in \mathcal{KLR}[c]$. Assume $M$ is lowest weight for the $\mathfrak{g}_c$ crystal structure, and $N$ is in the component generated by the trivial
representation. Then $M \circ N = N \circ M$, this module is irreducible, and $N \mapsto M \circ N$ is a bicrystal isomorphism between the component of the trivial representation and that of $M$.

**Lemma 3.24** With the notation of Proposition 3.23, $M \circ N$ has a unique simple quotient, and the map $N \mapsto A(M, N)$ commutes with crystal operators.

**Proof.** For any list of weights $v_1, \ldots, v_m$, let $e_{v_1, \ldots, v_m}$ be the idempotent that projects to all sequences which consist of a block of strands summing to $v_1$, a block summing to $v_2$, etc.

Choose any infinite list of nodes $j_1, j_2, \ldots$ in the Dynkin diagram of $\mathfrak{g}_c$ in which each node appears infinitely many times (for example, we can pick an order and cycle). In the crystal $B^k(-\infty)$, the module $N$ has a string parameterization $N = \tilde{e}_{j_1}^a \tilde{e}_{j_2}^a \cdots \tilde{e}_{j_\ell}^a L_0$, where $a_1$ is maximal number of times $\tilde{f}_{j_1}$ can be applied to $N$ before it becomes 0, $a_2$ the maximal number of times $\tilde{f}_{j_2}$ can be applied to $\tilde{f}_{j_1}^a N$, etc.

For any Lustzig datum $a$, define $e_a = e_{a, \beta_1, \ldots, \beta_\ell}$. Let $L_a$ denote the quotient of $L^a_{j_\ell} \circ L^a_{j_\ell-1} \circ \cdots \circ L^a_{j_1}$ by all $e_a' (L^a_{j_\ell} \circ L^a_{j_\ell-1} \circ \cdots \circ L^a_{j_1})$ for $a' > a$ in lexicographic order. By the definition of string parameterization, $N$ is a quotient of $L_a$.

Each of the roots $\beta_j$ is lowest, so $L^a_{j_\ell-1}$ is necessarily cuspidal for $c$, not just semicuspidal. Thus, the space $e_a L_a$ is spanned by diagrams which permute the simple terms of the induction. But any diagram that gives a non-trivial permutation must factor through the image of an idempotent $e_{a'}$ which we have killed (compare with the argument in [KL09, 3.7]). Thus $e_a L_a$ is just a copy of $L^a_{j_\ell} \boxtimes \cdots \boxtimes L^a_{j_1}$, which is simple. Now, we use the standard argument to show that $L_a$ has unique simple quotient—any proper submodule is killed by $e_a$, so their sum is as well, and thus is still proper.

Since $M$ is lowest weight for the $\mathfrak{g}_c$-crystal structure, $e_{wt(M) - \beta_1, \beta_2} M = 0$ for all $j$. The same argument as above then shows that $e_{h_\ell a} (M \circ L_a)$ is simple, and therefore $M \circ L_a$ has a unique simple quotient as well. But $M \circ N$ is a quotient of $M \circ L_a$, so it also has a unique simple quotient.

It is also clear that $\tilde{e}_{j_1}^a \tilde{e}_{j_2}^a \cdots \tilde{e}_{j_\ell}^a M$ is a quotient of $M \circ L_a$, so this must agree with the unique simple quotient of $M \circ N$, and hence the map $N \mapsto A(M, N)$ commutes with the ordinary crystal operators. \hfill $\square$

**Remark 3.25** The reader may notice the resemblance of the above argument to that we used earlier based on the unmixing property; unfortunately, neither $(M, L_a)$ nor $(M, L^a_{j_\ell}, \ldots, L^a_{j_1})$ is actually unmixing, so we must use this slightly more elaborate argument.
Proof of Proposition 3.23. The character of $M \circ N$ is the shuffle of the characters of $M$ and $N$. Since $M \boxtimes N$ is simple and is killed in no simple quotient, it is contained in the cosocle (which by Lemma 3.24 is simple). By [LV11, 2.2], the induction $M \circ N$ is isomorphic to the coinduction $\text{coind}(N \boxtimes M)$, so there is an injection from $N \boxtimes M$ into the socle of $M \circ N$.

If $i$ is a non-trivial word in the character of $M$, then the weight of any prefix $i'$ is either $< \delta$ or is a multiple of $\delta$. In particular, any word in the character of $M \circ N$ which begins with blocks that step along $\beta_{i_1}, \ldots, \beta_{i_d}$ for an arbitrary sequence $i_1, \ldots, i_d$, where $d = \sum a_i$ is the depth of $N$, must lie in the socle, since the multiplicities of such sequences is the same for $N \boxtimes M$ and $M \circ N$.

On the other hand, the cosocle is obtained from $M$ by applying $d$ crystal operators to $N$, so it can also be obtained by applying $d$ starred crystal operations by Corollary 3.22. Thus, there is at least one word in the character of the cosocle which begins with $d = \sum a_i$ many steps along $\beta_i$ for various $i$. It follows that the natural map from the socle to the cosocle must be non-zero, thus an isomorphism. Thus $M \circ N$ is in fact simple.

Notice also that the natural map from $N \circ M$ to the socle of $M \circ N$ must be non-zero and thus an isomorphism. Hence $N \circ M \simeq M \circ N$.

We have already established that $N \to A(M,N) = M \circ N$ is a crystal isomorphism for the unstarred operators; the symmetric argument for $N \circ M$ establishes that it is for the starred operators as well. □

3.3. Affine polytopes. Outside of finite type, the conventional definition of MV polytope fails, although as shown in [BKT], an alternate geometric definition can be extended to symmetric affine type. We propose to use the decorated polytopes $\tilde{P}_L$ as the “general type MV polytopes.” This construction is not completely combinatorial, as the decoration consists of various representations of KLR algebras. However, in affine type, we can extract purely combinatorial objects.

For the rest of this section fix $\mathfrak{g}$ of affine type with rank $r + 1$. As usual, label the simple roots of $\mathfrak{g}$ by $\alpha_0, \ldots, \alpha_r$ with $\alpha_0$ being the distinguished vertex as in [Kac90]. We first prove some technical results concerning the structure of the semi-cuspidal representations of KLR algebras with weight a multiple of $\delta$. These will allow us to precisely define the partitions $\pi^\gamma$ associated with a simple $L$ in the introduction.

Consider the projection $p: \alpha_i \to \bar{\alpha}_i$ for $i \neq 0$, $\delta \to 0$ from affine root space to the root space for $\mathfrak{g}_{\text{fin}}$, the Lie algebra attached to the Dynkin diagram with the 0 node removed, where we use $\bar{\alpha}$ to denote roots in the finite type root system. In all cases other then $A_{2n}^{(2)}$, the image of this map is exactly the set of finite type roots along with 0 (this can be seen by checking that $p$ sends the simple affine roots to a set of finite type roots including all the simples, and using the affine Weyl group). For $A_{2n}^{(2)}$, the image also contains $\alpha/2$ for each of the long roots $\alpha$ in the finite type root system.
For each chamber coweight \( \gamma = \theta \omega \gamma \) in the finite type root system (i.e. each element in the Weyl group orbit of a fundamental coweight), define a charge \( c_\gamma \) by

\[
c_\gamma(\alpha) = \langle \gamma, p(\alpha) \rangle + i \rho(\alpha).
\]

The set of roots with argument \( \pi/2 \) for \( c_\gamma \) is a rank \( r \) affine sub-root system. Hence, for any \( L \), \( c_\gamma \) defines a vertical face of \( P_L \) and this is generically dimension \( r \).

Let \( \Delta_{\text{fin}} \) be the root system for \( g_{\text{fin}} \) and let \( \Delta_{\text{fin}}' \) be the sub-root-system of \( \Delta_{\text{fin}} \) on which \( \gamma \) vanishes. Fix a basis \( \Pi = \{ \eta_1, \ldots, \eta_r \} \) for \( \Delta_{\text{fin}}' \). There is a unique \( \eta_r \in \Delta_{\text{fin}} \) whose addition makes this a base of \( \Delta_{\text{fin}} \) and such that \( \langle \gamma, \eta_r \rangle = 1 \). Explicitly, \( \eta_r \) is the unique root with \( \langle \gamma, \eta_r \rangle = 1 \) such that \( \eta_r - \eta_i \) is never a root.

Let \( c_{\Pi} \) be a charge such that the roots sent to \( \pi/2 \) are exactly the linear combinations of \( p^{-1}(\eta_i) \) and \( \delta_i \), and such that, for all \( 1 \leq i \leq r-1 \), the positive roots in \( p^{-1}(\eta_i) \) are \( > c_{\Pi} \delta_i \). Clearly \( g_{c_{\Pi}} \) is rank 2 affine, and thus is of type \( A_1^{(1)} \) or \( A_2^{(2)} \). The positive cone for \( g \) defines simple roots for \( g_{c_{\Pi}} \), which we denote by \( \beta_1 \) and \( \beta_2 \). We always choose the labeling so that \( \langle \gamma, p(\beta_1) \rangle < 0 \) and thus \( \beta_1 > \beta_2 \).

For \( i = 0, 1 \), define \( \ell_i = \frac{|\eta_i|}{\sqrt{2}} \) (which is always 1 or 2).

**Definition 3.26** Let \( \mathcal{L}_{\lambda, \gamma} \) be the element of the lowest weight crystal for \( g_{\text{fin}} \) generated by the trivial module with Lusztig datum \( \lambda \) for the ordering \( \beta_1 > \beta_2 \) as defined in [BDKT]. Explicitly, one can easily show using the combinatorics in [BDKT] that

\[
\mathcal{L}_{\lambda, \gamma} = e_{\gamma}^{\ell_2}(e_{\gamma}^{\ell_1})(e_0^{\ell_1})(e_0^{\ell_2})(e_1^{\ell_1})(e_1^{\ell_2})(e_2)(e_2^*) \cdots \mathcal{L}_0.
\]

**Lemma 3.27** If \( M \) is a simple module of weight \( n \delta \) which is semi-cuspidal for both \( c_\gamma \) and \( c_{\Pi} \) and which is in the crystal component of \( \mathcal{L}_0 \) for \( c_{\Pi} \), then \( M \cong \mathcal{L}_{\lambda, \gamma} \) for some \( \lambda \).

**Proof.** As in the proof of Proposition 3.17, we can use Saito reflections to reduce to the case where \( \beta_2 \) is a simple root.

Consider a representation \( M \) which is \( c_{\Pi} \)-cuspidal of weight \( n \delta \). By Theorem 1.26 (see also Remark 1.27), \( M \) is of the form \( \mathcal{L}_{\lambda, \gamma} \) if and only if its real Lusztig data \( a_{(m+1)\delta_0 + m\beta_1}(M) \) (as defined in Definition 1.11) with respect to the order \( \beta_1 > \beta_2 \) is always trivial. Thus it suffices to prove that if \( M \) is semi-cuspidal and in the component of \( \mathcal{L}_0 \) for \( c_{\Pi} \), and \( M \) has non-trivial Lusztig data of the form \( a_{(m+1)\delta_0 + m\beta_1}(M) \) for some \( m \geq 0 \), then \( M \) is not semi-cuspidal for \( c_\gamma \).

We proceed by induction on the smallest integer \( m \) such that \( a_{(m+1)\delta_0 + m\beta_1}(M) \neq 0 \), proving the statement for all \( \gamma \) simultaneously. If \( m = 0 \) the statement is clear, giving the base case of the induction.

So assume \( m > 0 \). Consider \( \sigma_0^*M \). By Corollary 2.22, this must be semi-cuspidal for the charge \( c_{\Pi}^0 \). The face-crystal \( \mathcal{KQ}[c_{\Pi}^0] \) is still rank-2 affine, with simple roots \( \beta_2^0 \) and \( \beta_1^0 \), and the Lusztig data of \( \sigma_0^*M \) for the order \( \beta_1 < \beta_2^0 \) are given by \( a_\sigma(\sigma_0^*M) = a_{\sigma_0^0}(M) \)
for $\alpha \neq \beta_0$. Note that
\[ s_0((m + 1)\beta_0 + m\beta_1) = (m - 1)\beta_0 + m\beta_1. \]

Since our inductive assumption covered all chamber weights, we are assuming that
$\sigma_0^* M$ is not semi-cuspidal for $c_{s_0}$. But then applying Corollary 2.22 again it is clear
that $M$ is not semi-cuspidal for $c_\gamma$. This completes the proof. \qed

**Proposition 3.28** The modules $L_\pi; \gamma$ are a complete, irredundant list of lowest-weight
semi-cuspidal modules of argument $\pi/2$ for $c_\gamma$, and this labeling is independent of
the choice of base in $g_{c_\gamma}$.

**Lemma 3.29** Proposition 3.28 holds when $\gamma = \omega_\nu^i$ is a fundamental coweight, and
the base $\Pi = \{\eta_i\}$ is given by the simple roots excluding $\alpha_i$.

**Proof.** In this case, the lowest-weight semi-cuspidal modules of argument $\pi/2$ for $c_\omega^i$
are precisely the semi-cuspidal modules of argument $\pi/2$ which are killed by $\tilde{f}_j$ for
$j \neq i, 0$.

Now, consider a an irreducible representation $L$ which is semi-cuspidal of argument
$\pi/2$ and lowest-weight in $KLR[c_{\omega_i}]$. If $L$ is were not $c_{\Pi}$-semi-cuspidal, then there must
be a $c_{\Pi}$-cuspidal $Q$ whose weight is a real root $\alpha < c_{\Pi} \delta$ such that $L$ is a quotient of
$Q' \circ Q$ for some simple $Q'$. Since $\alpha < c_{\Pi} \delta$, $p(\alpha)$ is a positive multiple of a negative root
in $g_{\text{fin}}$, and so $\alpha \leq c_{\omega_i}^\nu \delta$. Since $L$ is $c_{\omega_i}^\nu$-semi-cuspidal, we must have that $\alpha \geq c_{\omega_i}^\nu \delta$, so
$\alpha = c_{\omega_i}^\nu \delta$, or equivalently $\alpha$ has argument $\pi/2$ for $c_{\omega_i}^\nu$. Since $L$ is $c_{\omega_i}^\nu$-semi-cuspidal,
this implies $Q$ is $c_{\omega_i}^\nu$-semi-cuspidal as well, and since $L$ is lowest weight for $g_{c_{\omega_i}^\nu}$ so is
$Q$; this is impossible if $\alpha$ is a real root by Corollary 3.22. Thus, $L$ is $c_{\Pi}$-semi-cuspidal.

As in Proposition 3.23, there exist canonical representations with $M$ lowest-weight
and $N$ in the component of the identity for $g_{c_{\Pi}}$ such that $L = M \circ N = N \circ M$. Thus,
we must have that $M$ and $N$ are both semi-cuspidal and lowest-weight for $g_{c_{\omega_i}^\nu}$. The
representation $M$ is killed
- by $f_i^+$ since it is lowest-weight in $KLR[c_{\Pi}]$,
- by $f_0^+$ since it is semi-cuspidal for $g$ and $\alpha_0$ is the lowest root for this order, and
- by all other $f_j^+$'s since it is lowest-weight for $KLR[c_{\omega_i}^\nu]$.

Thus, we must have $M = \mathcal{L}_\emptyset$, and $L = N$.

Since $L$ is semi-cuspidal for $c_{\omega_i}^\nu$, Lemma 3.27 implies that the Lusztig datum of $L$
for the action of $g_{c_{\Pi}}$ ordering $\beta_2 > \beta_0$ must be purely imaginary. Since the number of
lowest-weight cuspidals for $c_{\omega_i}^\nu$ of each weight is the same as the number of simples
in the component of the trivial with imaginary Lusztig data, these sets of simples
must coincide. This establishes the result for $\gamma = \omega_\nu^i$, with the obvious choice of
base. \qed

**Proof of Proposition 3.28.** We reduce all other cases to that covered in Lemma 3.29.
If $γ = α_1^γ$ but we have chosen a different base $Π' = \{η_i\}'$ of $π_{\text{fin},γ}$, then we can find an element $w = s_{i_1} \cdots s_{i_k}$ of the Weyl group $W_{\text{fin},γ}$ such that $wη_i = η_i'$. Applying the Saito reflections $σ_{i_1} \cdots σ_{i_k}$ to $L_{γ,π}$ leaves $L_{γ,π}$ unchanged (since it is killed by $f_{i_1}$ and $f_{i_k}$ and has weight a multiple of $δ$), and also sends $L_{π,γ}$ as defined using $[η_i]$ to $L_{π,γ}$, as defined using the $[η_i']$. Thus $L_{γ,π}$ is independent of this choice.

Now consider a general chamber coweight $γ$. If $γ$ is not a fundamental coweight, then there must be a simple root $α_i$ for $i > 0$ such that $⟨γ, p(α_i)⟩ < 0$, so that the argument of $α_i$ with respect to $c_γ$ is greater than $π/2$. Notice that $α_i \neq β_2$, since $⟨γ, p(β_2)⟩ > 0$ Thus $φ_j(L_{π,γ}) = 0$, we can apply $σ_i$. If $α_i \neq β_2$ then by Lemma 3.18 applying $σ_i$ to all cuspidal modules for $c_γ$ defines an isomorphism of crystals to the same set-up for $c_{γ,y}$, which is negative on one fewer positive root in the finite type system than $γ$; in particular it sends $L_{π,γ}$ to $L_{π,β_2,γ}$. If $α_i = β_1$, the same fact follows from the known action of Saito reflections on $B(∞)$ for affine rank $2$ Lie algebras.

By induction, we may reduce to the case where $γ$ is a fundamental coweight, so the result follows by Lemma 3.29.

Fix a generic charge $c$ such that $c(δ) \in i\mathbb{R}_{>0}$. This defines a positive system in the finite type root system, where we say $α$ is positive if $p^{-1}(α) > δ$. Let $X_1, \ldots, X_r$ be the corresponding set of simple roots and $γ_1, \ldots, γ_r$ the dual set of coweights. For each $r$-tuples of partitions $π = (π_1, \ldots, π_r)$, define

$$L(π) = L_{π_1,γ_1} \circ L_{π_2,γ_2} \circ \cdots \circ L_{π_r,γ_r}. \tag{8}$$

\begin{remark} These agree with Kleshchev’s imaginary modules [Kle, §4.3]. Note that in contrast to Kleshchev, we have a canonical labeling of these by multipartitions.
\end{remark}

\begin{lemma} The module $L(π)$ is irreducible and independent of the ordering on simple roots. As $π$ ranges over multipartitions with $n$ boxes, these modules are all distinct and a complete list of $c$-semi-cuspidal representations of $R(nδ)$.
\end{lemma}

\begin{proof} We induct on the largest $j$ such that $π^{γ_j} \neq ∅$. Note that for the face defined by the common maxima of $γ_k$ for $1, \ldots, j - 1$, the module $L_{π^{γ_1},γ_1} \circ L_{π^{γ_2},γ_2} \circ \cdots \circ L_{π^{γ_{j-1}},γ_{j-1}}$ is a lowest weight simple for the crystal of the face by the inductive hypothesis. Furthermore, by definition, $L_{π^{γ_j},γ_j}$ is in the component of the identity for the face crystal. Thus, $L_{π^{γ_1},γ_1} \circ L_{π^{γ_2},γ_2} \circ \cdots \circ L_{π^{γ_j},γ_j}$ is irreducible by Proposition 3.23. The independence of ordering immediately follows from the irreducibility.

Clearly all the representations $L(π)$ are semi-cuspidal. They are all distinct, since the partition $π^{γ_i}$ is uniquely determined by the isomorphism type of $L(π)$. By Corollary 2.9, this is the right number of semi-cuspidal representations.
\end{proof}

Thus, if we consider the KLR polytope $P_L$ of $L$, the labeling of each real edge is determined by its length and direction; any imaginary edge parallel to $δ$ could be labeled with any $L(π)$ (where we fix a charge $c$ chosen so that the imaginary edge

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lies in its associated walk). However, we now show that this information can be encoded in a labeling of facets rather than edges.

Fix a simple \( L \) and a finite type chamber coweight \( \gamma \). Consider the semi-cuspidal decomposition \((\ldots, L_2, L_1, L_0, L_1^1, L_2, \ldots, )\) of \( L \) for \( c_\gamma \) where \( L_0 \) is the component with argument \( \pi/2 \).

**Definition 3.32** Let \( \pi^\gamma \) be the partition such that the representation \( L_0 \) lies in the crystal component of \( \mathcal{L}_{\pi^\gamma} \).

**Proposition 3.33** The representation decorating any imaginary edge \( E \) is exactly \( L(\pi^\gamma_1, \ldots, \pi^\gamma_r) \), where the \( \pi^\gamma_i \) are the chamber coweights which achieve their lowest value on \( E \), and \( \pi^\gamma \) is the partition associated to \( L \) and \( \gamma \) by Definition 3.32.

**Proof.** Fix a representation \( L \). Let \( c \) be a generic charge such that \( E \) is part of the path \( P^c_\gamma \). Let \( M \) be the representation in the semi-cuspidal decomposition of \( L \) for \( c \) whose weight is a multiple of \( \delta \). Then \( M \) is also semi-cuspidal for each \( c_\gamma \) (since the value of \( \gamma \) on the imaginary edge is a minimum on \( P^c_\gamma \)). We need only show that if \( M = \mathcal{L}(\xi^{\gamma_1}, \ldots, \xi^{\gamma_r}) \), then the partition \( \pi^{\gamma_i} \) attached to \( L \) by Definition 3.32 is \( \xi^{\gamma_i} \). If \( c \) is a small deformation of \( c_\gamma \), then is clear. Thus, we need only consider how \( \pi^\gamma \) changes as we deform \( c \) to \( c_\gamma \) linearly. While the semi-cuspidal decomposition changes, the imaginary edge that the associated walk goes along never changes. Thus, the representation in the semi-cuspidal decomposition of weight which is a multiple of \( \delta \) never changes. Thus, that representation is always \( M \). This establishes the result. □

Thus, the KLR polytope in the sense of Definition 3.4 can be encoded as a decorated affine pseudo-Weyl polytope as defined in the introduction, where we decorate the facet where \( \gamma \) achieves its minimum with \( \pi^\gamma \).

**3.4. An example.** If one were trying to naively generalize the finite type situation, it would be natural to hope that for a fixed generic charge, one could find a totally ordered set of cuspidal simples, with the number in each weight being the root multiplicity, such that, for \( L_1 \leq \cdots \leq L_k \) the module \( A(L_1^{n_1}, \ldots, L_k^{n_k}) \) gives a complete list of the simples. In fact, even in affine type, this will fail, as we now illustrate. We note that this example is also treated in [Kas, Example 3.3], but we wish to give a treatment emphasizing the features of interest to us.

We consider the case of \( \tilde{sl}_2 \), and choose a charge where \( \alpha_0 < c_1 \). We must also choose the polynomial \( Q_{01}(u,v) \), which we take to be \( u^2 + quv + v^2 \) for some \( q \in k \) (this is not a completely general choice of \( Q \), but any choice of \( Q \) gives an algebra isomorphic this one after passing to a finite field extension).

There are exactly two semi-cuspidal representations of weight \( 2\delta \). These can be described as \( \mathcal{L}_{(2),20} = \tilde{e}_1^2 \tilde{e}_0^2 \mathcal{L}_\emptyset \) and \( \mathcal{L}_{(1,1),0} = \tilde{e}_1 \tilde{e}_0 \tilde{e}_1 \mathcal{L}_\emptyset \). Consider the induction
$L_{(1)\omega} \circ L_{(1)\omega}$. This is 6-dimensional, spanned by the elements

$v = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$ \quad $\psi_2 v = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$ \quad $\psi_3 \psi_2 v = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$

$\psi_1 \psi_2 v = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$ \quad $\psi_3 \psi_1 \psi_2 v = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$ \quad $\psi_2 \psi_3 \psi_1 \psi_2 v = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$

where $v$ is any non-zero element of $L_{(1)\omega} \otimes L_{(1)\omega}$, which is 1-dimensional.

The span $H$ of the basis vectors other than $v$ is a submodule (it is the kernel of a map to $L_{(1,1)\omega}$). The image of the idempotent $e_{0011}$ is irreducible over $R(2\alpha_0) \otimes R(2\alpha_1)$, and generates $H$. Thus, either

- $H$ is irreducible or
- $\psi_2 \psi_3 \psi_1 \psi_2 v$ spans a submodule.

But,

$\psi_2^2 \psi_3 \psi_1 \psi_2 v = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} = q \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} = -q \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$

Thus, if $q \neq 0$, $H$ is irreducible and thus $H \cong L_{(2)\omega}$. Its inclusion is split, with complement spanned by $qv + \psi_2 \psi_3 \psi_1 \psi_2 v$. In particular, $L_{(1)\omega} \circ L_{(1)\omega}$ is semi-simple with both $L_{(2)\omega}$ and $L_{(1,1)\omega}$ occurring as summands. We see that neither of these modules can thus be cuspidal, since

$\text{ch}(L_{(2)\omega}) = (q^2 + 2 + q^2)w[0011] + w[0101]$. 

If $q = 0$, then the behavior is quite different; in this case $\psi_2 \psi_3 \psi_1 \psi_2 v$ spans the socle of $L_{(1)\omega} \circ L_{(1)\omega}$, and $H$ is its radical. In particular, $L_{(1)\omega} \circ L_{(1)\omega}$ is indecomposable, and a 3-step extension where a copy of $L_{(2)\omega}$ is sandwiched between the socle and cosocle, both isomorphic to $L_{(1,1)\omega}$. So in particular, when $q = 0$, the representation $L_{(2)\omega}$ is cuspidal, since

$\text{ch}(L_{(2)\omega}) = (q^2 + 2 + q^2)w[0011]$. 

The KLR polytopes of these representations are independent of $q$ and are given by
If one takes the choice of parameters as in [VV11] corresponding to an Ext-algebra of perverse sheaves on the moduli of representations of a Kronecker quiver (which is also that fixed by [BK09] in order to find a relationship to affine Hecke algebras with $\nu = -1$ or in characteristic 2), then we take $q = -2$. Thus, if the field $\mathbb{k}$ has characteristic $\neq 2$, we have $q \neq 0$ and $\dim L(2) = 5$ whereas if $\mathbb{k}$ does have characteristic 2, then $q = 0$ and $\dim L(2) = 4$. Under Brundan and Kleshchev’s isomorphism [BK09] between quotients of KLR algebras and cyclotomic Hecke algebras, this corresponds to the change in characters as we pass from the Hecke algebra at a root of unity to the symmetric group, or the difference between the canonical basis and 2-canonical basis.

In the $q = 0$ case, the number of cuspidals in this example is in fact the root multiplicity of $2\delta$. One might naively hope that at $q = 0$ this happened more generally, but explicit calculation in more complicated examples show that it does not.

3.5. Proof of Theorem B and Theorem C. Fix any convex order $\succ$. For each affine Lusztig datum there can be at most one decorated polytope satisfying the conditions of Theorem B which has the specified Lusztig datum with respect to $\succ$. Furthermore, by Theorem 2.3 and Corollary 2.9 we can always find a simple $L$ such that $P_L$ has this Lusztig datum. Thus, to prove Theorem B it suffices to show that each $P_L$ satisfies all the specified conditions on 2-faces.

Every 2-face is either real or parallel to $\delta$. The real 2-faces are themselves MV polytopes by Proposition 3.14. Thus it remains to check that on 2-faces parallel to $\delta$ also yield affine MV polytopes (after shortening the imaginary edge). Fix a charge $c$ such that the roots sent to the imaginary line form a rank 2 sub-root system, and let $g_c$ be the associated rank 2 affine algebra. This defines a (possibly degenerate) 2-face $F_c$ of any $P_L$, and all imaginary 2-faces occur this way for some such $c$.

Let $\gamma_1, \ldots, \gamma_{r-1}$ be the $r-1$ finite type chamber weights which define facets of $P_L$ containing $F_c$ for all $L$, and $\gamma_+, \gamma_-$ the two chamber weights that define faces that intersect $F_c$ in vertical lines. If you deform $c$ a small amount, then it gives a complete order on roots, and picks out one of the two vertical edges of $F_c$. We can choose deformations $c_\pm$ such that the set of chamber weights associated with these charges are $\{\gamma_1, \ldots, \gamma_{r-1}, \gamma_\pm\}$. Let $\beta_0$ and $\beta_\perp$ be the simple roots parallel to $F_c$ with

$$\langle \gamma_+, \beta_0 \rangle > 0 > \langle \gamma_+, \beta_\perp \rangle \quad \langle \gamma_-, \beta_\perp \rangle > 0 > \langle \gamma_-, \beta_0 \rangle.$$

We use the notations $\mathcal{L}_c, \mathcal{L}_{c_\pi}, \mathcal{L}(\pi)$ for the cuspidal representations for $c_+$ and $\mathcal{L}_c, \mathcal{L}_{c_\pi}, \mathcal{L}(\pi)$ for $c_-$. Similarly, we use $a_\beta$ to denote the Lusztig data of a $g$-polytope with respect to $c_+$ or a $g_c$-polytope for the order where $\beta_\perp > \beta_0$ and $\tilde{a}_\beta$ for $c_-$ or the order where $\beta_0 > \beta_\perp$.

We define a map $L \mapsto P^F_L$ from the set of $c$-semi-cuspidal representations of weight parallel to $F$ to decorated pseudo-Weyl polytopes for $g_F$ by sending $L$ to the polytope...
Proposition 3.34 The polytope $P^F_L$ is the MV polytope (in the sense of [BDKT]) for the crystal element $N$.

Proof. We must check the conditions of Theorem 1.26.

(i) This is clear when $L$ is a lowest weight object for the crystal (in which case the weight is 0 on both sides), and it is also clear that this property is preserved by the $g_c$ crystal operators.

(ii) Using Lemma 2.24, we can find Saito reflections in $B(-\infty)$ which reduce us to the case where $\beta_0$ or $\beta_1$ is simple for $g$. Hence this is a consequence of a consequence of Proposition 3.11 and the form of the $\ast$ involution on $B(-\infty)$.

(iii) Again, using Lemma 2.24, we can assume that $\beta_0$ or $\beta_1$ is simple in $g$. Assuming $\beta_0$ is simple, it is clear that the Saito reflections in this root in $B^{g_c}(-\infty)$ are the restrictions of the corresponding reflections in the full crystal $B(-\infty)$. Hence the statements for these two reflections are a consequence of Corollary 2.22. To get the statements for the reflections in $\beta_1$ we instead use Saito reflections in $B^{g}(-\infty)$ to reduce this to a simple root.

(iv) By definition, $\mathcal{L}_{\lambda;\gamma} = \tilde{\mathcal{L}}_{1,\lambda;\gamma} = \tilde{\mathcal{E}}_{1,\lambda;\gamma}$. Since this crystal element has trivial real Lusztig data for $\beta_0 > \beta_1$ and by the definition (9), this means that $\tilde{a}_0(P^F_{\mathcal{L}_{\lambda;\gamma}}) = 0$. Thus, we see that $\tilde{a}_0(P^F_{\mathcal{L}_{\lambda;\gamma}}) = \ell_1\lambda_1$.

This establishes the final condition of Theorem 1.26. □

Proof of Theorem C. It follows from Lemmas 1.17 and 1.18 that the sets of decorated pseudo-Weyl polytopes $P_L$ and $HN_b$ are both uniquely determined by a single Lusztig datum and relations extending the tropical Plücker relations that determine the structure of 2-faces. Thus, the recursive nature of the relations for both polytopes means we only need to check that the KLR polytopes and HN polytopes coincide in the rank 2 case. For real 2-faces, this follows from the Theorem A and [BKT, §1.5] and for
affine 2-faces, this follows from the match (with a transpose) of the \( \tilde{sl}_2 \) MV polytopes defined of [BKT] and [BDKT], which is shown in [MT]. Thus, every KLR polytope is an HN polytope with its imaginary Lusztig data transposed. This determines some bijection \( B(-\infty) \to B(-\infty) \). This bijection preserves Lusztig data for every convex order, is thus a crystal isomorphism, and it therefore is the identity. □

3.6. Beyond affine type. In affine type, while we can have many different semi-cuspidal representations corresponding to an imaginary root, we still have considerable control over the structure of these representations, as the proceeding sections show. In particular, the additional structure in their polytopes can be captured in a straight-forward way by labeling facets with partitions.

In general, we expect that the structure of a 2-face should be controlled by the set of roots obtained by intersecting a 2-dimensional plane with \( \Delta \). If \( \mathfrak{g} \) is of finite type then this set is also a finite type root system and the 2-faces are finite type MV polytopes. In affine type, this intersection is a rank-2 affine root system, and 2-faces are essentially rank 2 affine MV polytopes. But because of the multiplicities, the sum of these root spaces is actually not an affine Lie algebra—rather, it is an infinite-rank Borcherds algebra whose Cartan matrix is obtained by adding infinitely many rows and columns of zeroes to the rank 2 affine matrix. The structure we have observed in the 2-faces, a crystal for the affine Lie algebra together with an infinite family of commuting operators seems, in fact, to be a manifestation of this larger algebra, as opposed to just the affine one.

Beyond affine type, when one intersects \( \Delta \) with a 2-plane, the resulting set of real roots will generate a root system of rank at most 2. However, if there is to be a generalization of Theorem B, it is probably necessary to consider not just this rank 2 root system, but rather the entire sum of the root spaces; by [Bor95, Theorem 1], this will always be a Borcherds algebra. The corresponding Cartan matrix may have many non-zero entries outside the Cartan matrix of the root system generated by the real roots. Nonetheless, one could hope to define MV polytopes for this algebra, and that the 2-faces could be matched to these. Unfortunately, even if this were possible, “reduction to rank two” would mean reduction to a Borcherds algebra of possibly infinite rank, leaving it debatable whether this actually improves matters; it still may shed some rather interesting light on the structure of KLR algebras and their simple representations. Some cases, such as toroidal algebras (where the Cartan matrix remains positive semi-definite) may be more tractable.

To illustrate some of the difficulties, consider the Cartan matrix

\[
\begin{pmatrix}
2 & -2 & -2 \\
-2 & 2 & -2 \\
-2 & -2 & 2
\end{pmatrix}
\]

This is of hyperbolic type, and the imaginary root \( \beta = \alpha_1 + \alpha_2 + \alpha_3 \) has multiplicity 2. Fix a charge \( c \) with \( c(\alpha_0) = 1 + i, c(\alpha_1) = -1 + i, c(\alpha_2) = i \). The only real root with
$c(\alpha) \in i\mathbb{R}$ is $\alpha_2$ itself. Thus the real roots only generate a copy of $\mathfrak{sl}_2$, which is already a new phenomenon as in finite and affine type the real roots corresponding to a 2-face always generated a rank 2 root system.

Nonetheless, Proposition 3.17 shows that the semi-cuspidals of argument $\pi/2$ are a combinatorial crystal for $\mathfrak{sl}_2$. If the analogue of Corollary 3.22 held, then we would have that $\tilde{e}_2$ and $\tilde{e}_2^*$ were identical acting on every semi-cuspidal of argument $\pi/2$, since this is the case in $B_{\mathfrak{sl}_2}(-\infty)$. However, both $\tilde{e}_2\tilde{e}_1\tilde{e}_0\mathcal{L}_\emptyset$ and $\tilde{e}_2^*\tilde{e}_1\tilde{e}_0\mathcal{L}_\emptyset$ are 1-dimensional; the former has character $w[012]$ and the latter $w[201]$. Thus, they are necessarily distinct.

Attacking this case will require stronger techniques than we possess at the moment; the sharp-eyed reader will note that we give no direct connection between the KLR algebra attached to a face and the lower rank KLR algebra for the root system spanned by that face. While this seems like an obvious suggestion, we see no such connection (say, a functor) at the moment, and our techniques were intended to circumvent this absence. Perhaps more progress can be made on the hyperbolic case if such a functor can be found, using the KLR algebras for Borcherds algebras given in [KOP].

References


