KLR algebras and knot homology II

Ben Webster

University of Virginia

June 26, 2013
The case of $\mathfrak{sl}_2$

So, remember, last time I introduced an algebra $T^\ell$ attached to the $\mathfrak{sl}_2$ representation $(\mathbb{C}^2)^{\otimes \ell}$.

\[
\begin{align*}
\begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} & = \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} + \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} & \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} & = 0
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} & = \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} + \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} & \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} & = \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} + \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} & \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} & = 0
\end{align*}
\]

\[
\begin{align*}
\begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} & = \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} + \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} & \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} & = \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} + \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} & \begin{tikzpicture}[baseline, scale=0.5]
\draw[thick] (0,0) -- (0.5,0);
\draw[thick] (1,0) -- (1.5,0);
\draw[thick] (1.5,0) -- (2,0);
\end{tikzpicture} & = 0
\end{align*}
\]
Representations

One natural class of $T^\ell$ representations is the projectives (just fix sequence at the top, no relations):

$$P_{0,2} = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} \quad P_{1,0,2} = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array}$$

Under the isomorphism $K(T^\ell \text{-mod}) \cong (\mathbb{C}^2)^{\otimes \ell}$, we send

$$[P_{a_1,\ldots,a_\ell}] \rightarrow F^{a_\ell}([P_{a_1,\ldots,a_{\ell-1}}] \otimes v_+)$$

There are also standard modules $S_{a_1,\ldots,a_\ell}$, the quotient of $P_{a_1,\ldots,a_\ell}$ by the relations

$$\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} = 0$$

Under the isomorphism $K(T^\ell \text{-mod}) \cong (\mathbb{C}^2)^{\otimes \ell}$, we send

$$[S_{a_1,\ldots,a_\ell}] \rightarrow F^{a_1}v_+ \otimes \cdots \otimes F^{a_\ell}v_+$$
Let’s look again at the structure of the case of $T^2$ with one black line:

The only module we can see in this diagram killed by $\mathcal{C}$ is $L$. 
The case of $\mathfrak{sl}_2$

The cup (up to shift) is associated to derived tensor product with the bimodule:
The case of $\mathfrak{sl}_2$

The cup (up to shift) is associated to derived tensor product with the bimodule:

\[\text{Diagram of the bimodule} \]

Of course, we need to have some relations:

\[= 0\]

This is inserting two new tensor factors, and putting in the invariant module for those two factors.
The case of $\mathfrak{sl}_2$

The cup (up to shift) is associated to derived tensor product with the bimodule:
The case of $\mathfrak{sl}_2$

The cup (up to shift) is associated to derived tensor product with the bimodule:

Of course, we need to have some relations:

\[
\begin{align*}
\begin{tikzpicture}
\end{tikzpicture} &= 0 \\
\begin{tikzpicture}
\end{tikzpicture} &= 0 \\
\begin{tikzpicture}
\end{tikzpicture} &= 0 \\
\begin{tikzpicture}
\end{tikzpicture} &= 0
\end{align*}
\]

This is inserting two new tensor factors, and putting in the invariant module for those two factors.
The case of $\mathfrak{sl}_2$

So, for a circle, we get elements of the bimodule for each picture:

It’s easy to check that we can simplify so that the bubble is separate.

But we have to think a bit harder than this; the functor for a cup isn’t exact! You need to use a projective resolution!
The case of $\mathfrak{sl}_2$

So, for a circle, we get elements of the bimodule for each picture:

It’s easy to check that we can simplify so that the bubble is separate.

But we have to think a bit harder than this; the functor for a cup isn’t exact! You need to use a projective resolution!
The case of $\mathfrak{sl}_2$

So, for a circle, we get elements of the bimodule for each picture:

It’s easy to check that we can simplify so that the bubble is separate.

But we have to think a bit harder than this; the functor for a cup isn’t exact! You need to use a projective resolution!
The case of $\mathfrak{sl}_2$

We can evaluate compositions by noting that

\[
\begin{align*}
\begin{array}{ccc}
\text{Diagram} & = & \emptyset \\
\text{Diagram} & = & 0 \\
\text{Diagram} & = & 0 \\
\end{array}
\end{align*}
\]

So, we can see the relations

\[
\begin{array}{ccc}
-1 & 0 & 1 \\
\end{array}
\]
The case of $\mathfrak{sl}_2$

We can evaluate compositions by noting that

\[
\begin{align*}
\begin{array}{ccc}
\text{Diagram 1} & = & \emptyset \\
\text{Diagram 2} & = & 0 \\
\text{Diagram 3} & = & 0
\end{array}
\end{align*}
\]

So, we can see the relations

\[
\begin{align*}
\begin{array}{ccc}
\text{Diagram 4} & \xleftarrow{\psi} & \text{Diagram 5} \\
\text{Diagram 6} & \xleftarrow{\chi} & \text{Diagram 7} \\
\text{Diagram 8} & \xleftarrow{\chi} & \text{Diagram 9}
\end{array}
\end{align*}
\]
The case of $\mathfrak{sl}_2$

We can evaluate compositions by noting that

$$
\begin{align*}
&= \emptyset \\
&= 0 \\
&= 0
\end{align*}
$$

So, we can see the relations
The case of $\mathfrak{sl}_2$

We can evaluate compositions by noting that

\[ = \emptyset \quad = 0 \quad = 0 \]

So, we can see the relations

\[ -1 \quad 0 \quad 1 \]

\[ = \]

Ben Webster (UVA)  KLR algebras and knot homology II  June 26, 2013  7 / 25
The case of $\mathfrak{sl}_2$

We can evaluate compositions by noting that

\[ \emptyset = 0 \]
\[ \emptyset \oplus \emptyset = 0 \]

So, we can see the relations
The case of $\mathfrak{sl}_2$

We can evaluate compositions by noting that

\[
\begin{align*}
\text{diag} & = \emptyset \\
\text{cyclic} & = 0 \\
\text{anti-}\text{cyclic} & = 0
\end{align*}
\]

So, we can see the relations

\[
\begin{array}{cccc}
-1 & 0 & 1 \\
\emptyset & \oplus & \emptyset
\end{array}
\]
The action of cobordisms

**Theorem (Chatav)**

There is an action of the cobordism 2-category on the categories $D^b(T^n \text{-mod})$.

In fact, one can easily check that this action satisfies the Bar-Natan relations (S,T,G,NC).
The action of cobordisms

**Theorem (Chatav)**

*There is an action of the cobordism 2-category on the categories $D^b(T^n \text{-mod})$.*

In fact, one can easily check that this action satisfies the Bar-Natan relations (S,T,G,NC).
The action of cobordisms

Theorem (Chatav)

There is an action of the cobordism 2-category on the categories $D^b(T^n \text{-mod})$.

In fact, one can easily check that this action satisfies the Bar-Natan relations (S,T,G,NC).
The action of cobordisms

**Theorem (Chatav)**

There is an action of the cobordism 2-category on the categories $D^b(T^n\text{-mod})$.

In fact, one can easily check that this action satisfies the Bar-Natan relations (S,T,G,NC).
The action of cobordisms

Theorem (Chatav)

There is an action of the cobordism 2-category on the categories $D^b(T^n\text{-mod})$.

In fact, one can easily check that this action satisfies the Bar-Natan relations (S,T,G,NC).
The action of cobordisms

Fun game: look at a construction in BN world, transport it to modules over $T^\ell$.

Dror tells us that

\[
\begin{array}{c}
\times \\
= \text{Cone}\left(\begin{array}{c}
\uparrow \\
\downarrow \\
\end{array}\right) \left(\begin{array}{c}
\rightarrow \\
\end{array}\right)
\end{array}
\]

The cone of an surjective map of bimodules over $T^\ell$ is its kernel.

The kernel is the set of diagrams where we can see a pair of red lines with no black between them.
The action of cobordisms

Fun game: look at a construction in BN world, transport it to modules over $T^\ell$.

Dror tells us that

\[ \begin{array}{c} \times \end{array} = \text{Cone} \begin{array}{c} (\rightarrow) \end{array} \]

The cone of an surjective map of bimodules over $T^\ell$ is its kernel.

The kernel is the set of diagrams where we can see a pair of red lines with no black between them.

To remind that we have to go around, snap the red strands together.
Comparison to Khovanov homology

These bimodules define a functor $D^b(T^\ell_1\text{-mod}) \to D^b(T^\ell_2\text{-mod})$ for every tangle connecting $\ell_1$ points to $\ell_2$; I should say “tangle projection” but we already know isotopy invariance!

Theorem

The resulting knot invariant is Khovanov homology.

Note that we can make functoriality work as usual using the action of cobordisms. If you want to avoid sign problems, you need to use the Morrison-Walker disorientation scheme.
Cooper and Krushkal constructed a categorified JW projector in Bar-Natan’s category.

The image over $T^\ell$ is surprisingly easy to describe.

\[ \cdots \]

w/ relations

\[ \begin{align*}
\bullet a &= b \\
\end{align*} \]

You can think of this as realizing $T^{(n_1, \ldots, n_\ell)}$-mod as a quotient category of $T^{n_1 + \cdots + n_\ell}$-mod.

This allows us to transport their categorification of colored Jones to our picture. Actually, this is a very special case of the general construction.
So, let’s move on to the general case.

Let $T^\lambda$ be the algebra generated by diagrams of red and black lines; red lines are labeled by weights, black by simple roots.

- Red line labeled by weight $\lambda_j$:
  \[ \text{deg} = \langle \alpha_{ij}, \alpha_{ij} \rangle \]

- Red line labeled by weight $\lambda_{ij}$ and black line labeled by simple root $\alpha_{ij}$:
  \[ \text{deg} = -\langle \alpha_{ij}, \alpha_{ij+1} \rangle \]

- Red line labeled by weight $\lambda_{ij}$ and black line labeled by simple root $\lambda_k$:
  \[ \text{deg} = \langle \alpha_{ij}, \lambda_k \rangle \]
Relations

\[ i \otimes i = i \otimes i + i \otimes i \quad \otimes i = a_{ij} \otimes i + i \otimes a_{ji} \]

\[ i \otimes i = 0 \quad i \otimes i = i \otimes i + \sum_{a+b=a_{ij}-1} a \otimes i + b \]

\[ i \lambda \otimes \lambda = \alpha_i^\lambda(\lambda) \otimes \lambda + \sum_{a+b=a_i^\lambda(\lambda)-1} a \otimes \lambda + b \]

\[ \lambda \otimes i = \alpha_i^\lambda(\lambda) \otimes i \]

\[ \lambda \otimes \lambda = \alpha_i^\lambda(\lambda) \otimes i \]

\[ \cdots = 0 \]
The Grothendieck group of the category $C(\lambda) := D(T^\lambda \text{-mod})$ is the tensor product $V_\underline{\lambda} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell}$; the categorical $g$-action induces the usual action of $g$ on $V_\underline{\lambda}$.

So, I want to define a 4-d extended TQFT that assigns $C(\underline{\lambda})$ to a disk, and $\text{Inv}(C(\underline{\lambda}))$ to a sphere labeled with $\lambda_1, \ldots, \lambda_\ell$.

Now, we want to assign functors to cobordisms with Wilson lines (i.e. a tangle).
I already told you what these functors are, but let me tell you a little more carefully.

A positive crossing is sent to the bimodule $\mathcal{B}_j$ where the $j$th and $j + 1$st red lines cross exactly once. All relations involving the red-red crossing are naive isotopies.
Braid operators

Theorem

The functors

\[ - \otimes B_i : C(\lambda) \rightarrow C(s_i \cdot \lambda) \]

are equivalences that satisfy the braid relations.

The induced maps on Grothendieck groups \( V_\lambda \rightarrow V_\lambda \) is the braiding/R-matrix times flip/map on Hilbert spaces induced by cobordism.

This helps us address a nagging point, which is that the category depends on an ordering of the weights \( \lambda_i \), whereas on the surface they are allowed to move around.

The braid group is exactly the mapping class group of a punctured disk (fixing the boundary), so we expect it to act on the category \( C(\lambda) \).
Tangle invariants

Associated to a cup, we also have a functor. Like the $\mathfrak{sl}_2$ case, we just draw the cup in the red lines, but we to have something pop out when we hit the minimum.

Just as before, you have to “insert a copy” of the unique invariant simple. It’s a bit more unpleasant to say exactly what that means; basically you take black lines corresponding to going from the highest to lowest weight vector in $V_\mu$, and get 0 if pull a strand out of the cup.
Tangle invariants

Associated to a cup, we also have a functor. Like the $\mathfrak{sl}_2$ case, we just draw the cup in the red lines, but we to have something pop out when we hit the minimum.

\[ \mu \quad \rightarrow \quad \mu^* \]

\[ \cdots \]

Just as before, you have to “insert a copy” of the unique invariant simple. It’s a bit more unpleasant to say exactly what that means; basically you take black lines corresponding to going from the highest to lowest weight vector in $V_\mu$, and get 0 if pull a strand out of the cup.
Cups and caps

This picture and its mirror image define two bimodules.

**Theorem**

The functors attached to cups and caps categorify the (co)evaluation and quantum (co)trace maps on tensor products (that is, those associated to the Wilson operators).

These do satisfy the expected adjunctions, so we get an invariant of flat tangles; however, we don’t seem to get an action of cobordisms in general. We only get this when $\lambda_i$’s are all minuscule.

Outside the minuscule case, we won’t even get a finite dimensional invariant for a circle! The problem is that the invariant simple might not have a finite resolution.
For every ribbon tangle $\tau$ in $D^2 \times I$, we have a functor $F_\tau : C(\Lambda) \to C(\Lambda')$ between the categories associated to the top and bottom, such that $F_{\tau' \circ \tau} \cong F_{\tau'} \circ F_{\tau}$, which categorifies the Reshetikhin-Turaev construction.

In particular, we have a doubly graded invariant attached to links with components labeled by $\mathfrak{g}$-representations. The Euler characteristic of this homology is the invariant obtained from Chern-Simons theory.
The case of $\mathfrak{sl}_3$

For $\mathfrak{sl}_2$, I could rewrite everything in terms of Dror’s construction. For $\mathfrak{sl}_3$, it’s going to be slightly harder. In addition to the cups and caps, we need some trivalent vertices.

\[ \begin{array}{c}
2 \\
1 \quad 1 \\
1 \quad 1
\end{array} \quad \text{and} \quad \begin{array}{c}
1 \\
2 \quad 2 \\
2 \quad 2
\end{array} \]

The relations are quite similar to those of the cup:

\[ \begin{array}{c}
2 \\
1 \quad 1 \quad 1
\end{array} = \begin{array}{c}
2 \\
1 \quad 1 \quad 1
\end{array} = 0 \quad \text{and} \quad \begin{array}{c}
2 \\
1 \quad 1 \quad 1
\end{array} = \pm \begin{array}{c}
2 \\
1 \quad 1 \quad 1
\end{array} \]
The case of $\mathfrak{sl}_3$

For $\mathfrak{sl}_2$, I could rewrite everything in terms of Dror’s construction. For $\mathfrak{sl}_3$, it’s going to be slightly harder. In addition to the cups and caps, we need some trivalent vertices.

\[
\begin{align*}
2 & \ 1 & \ 1 \\
1 & \ 1 & \ 1 \\
\end{align*}
\quad =
\quad \begin{align*}
1 & \ 1 & \ 1 \\
2 & \ 2 & \ 2 \\
\end{align*}
\]

The relations are quite similar to those of the cup:

\[
\begin{align*}
2 & \ 1 & \ 1 & \ 1 \\
1 & \ 1 & \ 1 \\
\end{align*}
\quad =
\quad \begin{align*}
2 & \ 1 & \ 1 & \ 1 \\
1 & \ 1 & \ 1 \\
\end{align*}
\quad =
\quad \begin{align*}
2 & \ 1 & \ 1 & \ 1 \\
1 & \ 1 & \ 1 \\
\end{align*}
\quad =
\quad \begin{align*}
2 & \ 1 & \ 1 & \ 1 \\
1 & \ 1 & \ 1 \\
\end{align*}
\quad = \pm
\]
The case of $\mathfrak{sl}_3$

The same argument as in $\mathfrak{sl}_2$ shows that we have an exact sequence

$$
\begin{array}{cccccccc}
1 & 1 & \rightarrow & 1 & 1 & \rightarrow & 1 & 1 \\
1 & 1 & & 1 & 1 & & 1 & 1 \\
\end{array}
$$

Using adjunctions, one can also show that

$$
\begin{array}{cccccccc}
2 & 1 & \rightarrow & 2 & 1 & \rightarrow & 2 & 1 \\
1 & 2 & & 1 & 2 & & 1 & 2 \\
\end{array}
$$

These look pretty reminiscent of Khovanov’s approach to $\mathfrak{sl}_3$ homology.
Khovanov described an approach to categorifying the RT invariant for \( \mathbb{C}^3 \) as a module over \( \mathfrak{sl}_3 \) which is reminiscent of Dror’s picture, but harder. You have to allow cobordisms with sheets labeled by fundamental representations, and singularities like our bimodules, called \textbf{foams}.

**Theorem**

There’s an action of \( \mathfrak{sl}_3 \) foams on these bimodules, which matches the braiding bimodules with the expected complexes of foams.

In particular, the knot homologies we’ve constructed match the Khovanov \( \mathfrak{sl}_3 \) homology.
The case of $\mathfrak{sl}_n$

There are similar descriptions for $\mathfrak{sl}_n$, though considerably more complicated. You end up with resolutions like

$$
\begin{array}{c}
2 & 2 & 2 \\
\times & \rightarrow & 2 \\
2 & 2 & 2
\end{array}
\rightarrow
\begin{array}{c}
2 & 2 & 2 \\
\rightarrow & 1 & 3 \\
2 & 2 & 2
\end{array}
\rightarrow
\begin{array}{c}
2 & 2 & 2 \\
\rightarrow & 4 \\
2 & 2 & 2
\end{array}
$$

For experts: this is a Koszul dual categorical action of $\mathfrak{sl}_e$ categorifying skew Howe duality, and the braiding functors are exactly the Chuang-Rouquier braid functors for this dual action.
The case of $\mathfrak{sl}_n$

There is an already known invariant for the defining representation $\mathbb{C}^n$ of $\mathfrak{sl}_n$: **Khovanov-Rozansky homology** (with extensions to other fundamentals by Wu and Yonezawa).

One question that has bugged me for a long time is whether this matches the invariants already discussed. You can easily match my approach with earlier work of Mazorchuk and Stroppel in this case.

Based on recent work of Mackaay and Yonezawa, I can now assert:

**Theorem?**

*They match!*

I consider the question mark to be an excess of caution, but at the moment, the argument is unnecessarily circuitous, and should be made more direct.
Thanks for your attention.