

NEW PROOF OF THE CHEEGER-MÜLLER THEOREM

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ABSTRACT. We present a short analytic proof of the equality between the analytic and combinatorial torsion. We use the same approach as in the proof given by Burghelea, Friedlander and Kappeler, but avoid using the difficult Mayer-Vietoris type formula for the determinants of elliptic operators. Instead, we provide a direct way of analyzing the behaviour of the determinant of the Witten deformation of the Laplacian. In particular, we show that this determinant can be written as a sum of two terms, one of which has an asymptotic expansion with computable coefficients and the other is very simple (no zeta-function regularization is involved in its definition).

1. INTRODUCTION

1.1. **Cheeger-Müller theorem.** Let F be a flat vector bundle over a compact odd dimensional Riemannian manifold M . Suppose that F is equipped with a Hermitian metric g^F , which induces a flat metric on the determinant line bundle $\det F$. These data define the Ray-Singer metric $\|\cdot\|_{\det H^\bullet(M,F)}^{RS}$ on the determinant line $\det H^\bullet(M,F)$, cf. [BZ1, Def. 2.2] and Def. 3.3 of this paper.

Let $f : M \rightarrow \mathbb{R}$ be a Morse function satisfying the Thom-Smale transversality conditions, [Sm1, Sm2]. Then one defines the Milnor metric $\|\cdot\|_{\det H^\bullet(M,F)}^{\mathcal{M}}$ on $\det H^\bullet(M,F)$, cf. [BZ1, Def. 1.9] and Def. 2.6 of this paper.

Theorem 1.2. $\|\cdot\|_{\det H^\bullet(M,F)}^{RS} = \|\cdot\|_{\det H^\bullet(M,F)}^{\mathcal{M}}$.

For the case when the metric g^F is flat, the theorem was conjectured by Ray and Singer [RS]. The Ray-Singer conjecture was proven independently by Cheeger [Ch] and Müller [Mü1]. Later Müller [Mü2] extended the result to the case when g^F is not necessarily flat, but the induced metric on $\det F$ is flat. The methods of Cheeger and Müller are both based on a combination of the topological and analytical methods. Bismut and Zhang [BZ1] suggested a purely analytical proof of the Ray-Singer conjecture and generalized it to the case, when the dimension of M is not necessarily odd and the induced metric on $\det F$ is not flat (in the latter case the relationship between the Milnor and the Ray-Singer metric is more complicated).

Another purely analytical proof of Theorem 1.2 was suggested by Burghelea, Friedlander and Kappeler [BFK2]. Their method was based on application of the highly non-trivial Mayer-Vietoris-type formula for the determinant of an elliptic operator [BFK1].

In this paper we suggest a new proof of Theorem 1.2, which essentially follows the lines of [BFK2] but is considerably simpler in several steps. In particular, we avoid the use of Mayer-Vietoris-type formula from [BFK1].

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Note that our proof can be modified to cover the more complicated case of even-dimensional manifolds, considered in [BZ1]. However, in this note we restrict ourselves to odd dimensional manifolds, since our purpose is to illustrate the method on the simplest possible example.

1.3. The sketch of the proof. Let $d^F : \Omega^\bullet(M, F) \rightarrow \Omega^{\bullet+1}(M, F)$ be the de Rham differential. Consider the *Witten deformation* $d_t^F = e^{-tf} d^F e^{tf}$ of d^F and set $\Delta_{f,t} = d_t^F d_t^{F*} + d_t^{F*} d_t^F$. It was discovered by Witten [Wi] (see also [HS, CFKS, Sh2, BFKM]) that, when $t \rightarrow \infty$, finitely many eigenvalues of $\Delta_{f,t}$ tend to zero (these are, so called “small” eigenvalues), while the rest of the eigenvalues tend to infinity (these are “large” eigenvalues). The Ray-Singer metric can be expressed, roughly speaking, as the product of the contribution of the “small” eigenvalues and the contribution of the “large” eigenvalues, cf. Sect. 4.5. The proof is naturally divided into the study of those two contributions.

The contribution of the “small” eigenvalues is summarized in Theorem 4.6, which was proven by Bismut and Zhang [BZ1]. The original proof was based on difficult results of Helffer and Sjöstrand [HS]. Later Bismut and Zhang [BZ2, §6] found a short and very elegant proof of this result (see also [BFKM]).

It remains to study the contribution of the “large” eigenvalues, which we denote by $\rho_{la}^{RS}(f, t)$, cf. Sect. 4.5. Let $P_{la,t}$ denote the orthogonal projection on the span of the eigenforms corresponding to the “large” eigenvalues of $\Delta_{f,t}$. The contribution $\rho_{la}^{RS}(f, t)$ of the “large” eigenvalues is defined in terms of $\log \det'[\Delta_{f,t} P_{la,t}]$, where \det' denotes the regularized determinant, cf. Sect. 3.2. Our method is based on the following simple formula, cf. Prop. 5.7,

$$k \log \det'[\Delta_{f,t} P_{la,t}] = \log \det'[(\Delta_{f,t}^k + t^{2k}) P_{la,t}] - t^{2k} \int_0^1 \text{Tr}[(\Delta_{f,t}^k + \tau t^{2k})^{-1} P_{la,t}] d\tau. \quad (1.1)$$

Here we choose $k > n/2$, so that the operator $(\Delta_{f,t}^k + \tau t^{2k})^{-1}$ is in the trace class.

The first summand in the right hand side of (1.1) is the logarithm of the determinant of an *operator elliptic with parameter*, cf. [Sh1, BFK1]. It is shown in the Appendix to [BFK1] that it has a nice asymptotic expansion with computable coefficients. The second summand, though does not have an asymptotic expansion, is very simple since no ζ -function regularization is needed to define it. It is not difficult now to prove the following result (cf. Theorem 5.4): Let \widetilde{M} be another Riemannian manifold and $\widetilde{F} \rightarrow \widetilde{M}$ be a flat vector bundle over \widetilde{M} such that $\dim \widetilde{F} = \dim F$. Let $\widetilde{f} : \widetilde{M} \rightarrow \mathbb{R}$ be a Morse function. Assume that the functions f and \widetilde{f} have *the same critical points structure*, cf. Def. 5.2. Then $\log \rho_{la}^{RS}(f, t) - \log \rho_{la}^{RS}(\widetilde{f}, t)$ has a nice asymptotic expansion with computable coefficients. This result was central in the Burghelca-Friedlander-Kappeler proof, cf. Theorem B of [BFK2].

Set

$$R(M, F, f) := \log \frac{\|\cdot\|_{\det H^\bullet(M, F)}^{RS}}{\|\cdot\|_{\det H^\bullet(M, F)}^{\mathcal{M}}}. \quad (1.2)$$

By [Mi2, Th. 9.3], the Milnor metric, and, hence, $R(M, F, f)$ is independent of f . It is, however, convenient to keep f in the notation.

In Sect. 6, we show, that, if f and \tilde{f} have the same critical points structure, then $R(M, F, f) = R(\tilde{M}, \tilde{F}, \tilde{f})$. It follows from [Mi1] that there exist Morse functions f_1, f_2 satisfying the Thom-Smale condition on $M \times S^2$ and $M \times S^1 \times S^1$ respectively, which have the same critical points structure. Let F_1, F_2 denote the lifts of F to $M \times S^2$ and $M \times S^1 \times S^1$ respectively. Then

$$R(M \times S^2, F_1, f_1) = R(M \times S^1 \times S^1, F_2, f_2). \quad (1.3)$$

Theorem 2.5 of [RS] expresses the Ray-Singer torsion of the product $M \times N$ (here N is a compact Riemannian manifold) in terms of the Ray-Singer torsion of M . In Sect. 6, we use this result to show that

$$R(M, F, f) = R(M \times S^2, F_1, f_1), \quad \text{and} \quad R(M \times S^1 \times S^1, F_2, f_2) = 0. \quad (1.4)$$

Combining (1.3) and (1.4) we obtain $R(M, F, f) = 0$.

1.4. The results used in the proof. For convenience of the reader, we list all the results which we use but don't prove in this paper.

- The invariance of the Milnor and the Ray-Singer torsion under diffeomorphisms of M . The proofs can be found in [Mi2, Th. 9.3] and [RS, Th. 2.1] respectively.
- The relationship between the Milnor metric and the contribution of the “small” eigenvalues of the Witten deformation of the Laplacian to the Ray-Singer torsion, cf. Theorem 4.6. A very nice proof can be found in [BZ2, §6] (see also [BFKM]).
- The asymptotic expansion of the trace and the determinant of an operator elliptic with parameter obtained in the Appendix to [BFK1].
- Existence of a constant $C > 0$ such that $\text{Tr} [(\Delta_{f,t}^k + \varepsilon)^{-1} P_{la,t}] < C$ for all $k > n$, $\varepsilon > 0$ and $|t| \gg 0$. This simple estimate follows, for example, from Lemma 3.3 of [BFK2].
- The expression for Ray-Singer torsion on the product of 2 manifolds, cf. [RS, Th. 2.5].

Apart from these results the paper is completely self-contained.

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2. THE MILNOR METRIC AND THE MILNOR TORSION

2.1. The determinant line of a finite dimensional complex. If λ is a real line, let λ^{-1} be the dual line. If E is a finite dimensional vector space, set $\det E = \Lambda^{\max}(E)$. Let $(V^\bullet, \partial) : 0 \rightarrow V^0 \rightarrow \dots \rightarrow V^n \rightarrow 0$ be a complex of finite dimensional Euclidean vector spaces. Let $H^\bullet(V) = \bigoplus_{i=0}^n H^i(V)$ be the cohomology of (V^\bullet, ∂) . Set

$$\det V^\bullet = \bigotimes_{i=0}^n (\det V^i)^{(-1)^i}, \quad \det H^\bullet(V) = \bigotimes_{i=0}^n \left(\det H^i(V) \right)^{(-1)^i}.$$

Then, by [KM], there is a canonical isomorphism of real lines

$$\det H^\bullet(V) \simeq \det V^\bullet. \quad (2.1)$$

2.2. Two metrics on the determinant line. The Euclidean structure on V^\bullet defines a metric on $\det V^\bullet$. Let $\|\cdot\|_{\det H^\bullet(V)}$ be the metric on the line $\det H^\bullet(V)$ corresponding to this metric via the canonical isomorphism (2.1).

Let ∂^* be the adjoint of ∂ with respect to the Euclidean structure on V^\bullet . Using the finite dimensional Hodge theory, we have the canonical identification

$$H^i(V^\bullet, \partial) \simeq \{v \in V^i : \partial v = 0, \partial^* v = 0\}, \quad 0 \leq i \leq n. \quad (2.2)$$

As a vector subspace of V^i , the vector space in the right-hand side of (2.2) inherits the Euclidean metric. We denote by $|\cdot|_{\det H^\bullet(V)}$ the corresponding metric on $\det H^\bullet(V)$.

The metrics $\|\cdot\|_{\det H^\bullet(V)}$ and $|\cdot|_{\det H^\bullet(V)}$ do not coincide in general. We shall describe the discrepancy.

Set $\Delta = \partial\partial^* + \partial^*\partial$ and let Δ^i denote the restriction of Δ to V^i . Let $\det' \Delta^i$ denote the product of the non-zero eigenvalues of Δ^i .

Definition 2.3. *The torsion ρ of the complex (V^\bullet, d) is the number defined by the formula*

$$\log \rho = \frac{1}{2} \sum_{i=0}^n (-1)^i i \log \det' \Delta^i.$$

The following result is proved, e.g., in [BGS, Prop. 1.5]

$$\|\cdot\|_{\det H^\bullet(V)} = |\cdot|_{\det H^\bullet(V)} \cdot \rho.$$

2.4. The Thom-Smale complex. Let $f : M \rightarrow \mathbb{R}$ be a Morse function satisfying the Smale transversality conditions [Sm1, Sm2] (for any two critical points x and y of f the stable manifold $W^s(x)$ and the unstable manifold $W^u(y)$, with respect to ∇f , intersect transversely).

Let B be the set of critical points of f . If $x \in B$, let F_x denote the fiber of F over x and let $[W^u(x)]$ denote the real line generated by $W^u(x)$. For $0 \leq i \leq n$, set

$$C^i(W^u, F) = \bigoplus_{\substack{x \in B \\ \text{ind}(x)=i}} [W^u(x)]^* \otimes_{\mathbb{R}} F_x.$$

By a basic result of Thom ([Th1]) and Smale ([Sm2]) (see also [BZ1, pp. 28–30]), there are well defined linear operators

$$\partial : C^i(W^u, F) \rightarrow C^{i+1}(W^u, F),$$

such that the pair $(C^\bullet(W^u, F), \partial)$ is a complex and there is a canonical identification of \mathbb{Z} -graded vector spaces

$$H^\bullet(C^\bullet(W^u, F), \partial) \simeq H^\bullet(M, F). \quad (2.3)$$

2.5. The Milnor metric. By (2.1) and (2.3), we know that

$$\det H^\bullet(M, F) \simeq \det C^\bullet(W^u, F). \quad (2.4)$$

The metric g^F on F determines the structure of an Euclidean vector space on $C^\bullet(W^u, F)$. This structure induces a metric on $\det C^\bullet(W^u, F)$.

Definition 2.6. The Milnor metric $\|\cdot\|_{\det H^\bullet(M, F)}^{\mathcal{M}}$ on the line $\det H^\bullet(M, F)$ (cf. [BZ1, §Id]) is the metric corresponding to the above metric on $\det C^\bullet(W^u, F)$ via the canonical isomorphism (2.4).

By Milnor [Mi2, Th. 9.3], the Milnor metric coincides with the Reidemeister metric defined through a smooth triangulation of M . It follows that $\|\cdot\|_{\det H^\bullet(M, F)}^{\mathcal{M}}$ does not depend upon f and g^{TM} , g^F and, hence, is a topological invariant of the flat bundle F .

3. THE RAY-SINGER METRIC AND THE RAY-SINGER TORSION

3.1. The L^2 metric on the determinant line. Let $(\Omega^\bullet(M, F), d^F)$ be the de Rham complex of the smooth sections of $\Lambda(T^*M) \otimes F$ equipped with the coboundary operator d^F . The cohomology of this complex is canonically isomorphic to $H^\bullet(M, F)$.

Let $*$ be the Hodge operator associated to the metric g^{TM} . We equip $\Omega^\bullet(M, F)$ with the inner product

$$\langle \alpha, \alpha' \rangle_{\Omega^\bullet(M, F)} = \int_M \langle \alpha \wedge * \alpha' \rangle_{g^F}. \quad (3.1)$$

By Hodge theory, we can identify $H^\bullet(M, F)$ with the space of harmonic forms in $\Omega^\bullet(M, F)$. This space inherits the Euclidean product (3.1). The L^2 -metric $|\cdot|_{\det H^\bullet(M, F)}^{RS}$ on $\det H^\bullet(M, F)$ is the metric induced by this product.

3.2. The Ray-Singer torsion. Let d^{F*} be the formal adjoint of d^F with respect to the metrics g^{TM} and g^F . Let $\Delta = d^F d^{F*} + d^{F*} d^F$ be the Laplacian and let Δ^i denote the restriction of Δ to $\Omega^i(M, F)$. Let $P^i : \Omega^i(M, F) \rightarrow \text{Ker } \Delta^i$ be the orthogonal projection.

To define the torsion of the complex $(\Omega^\bullet(M, F), d^F)$ one needs to make sense of the notion of determinant of the Laplacian. This is done using the zeta-function regularization as follows.

For $s \in \mathbb{C}$, $\text{Re } s > n/2$, set $\zeta_i^{RS}(s) = -\text{Tr} [(\Delta^i)^{-s} (I - P^i)]$. By a result of Seeley [Se], $\zeta_i^{RS}(s)$ extends to a meromorphic function of $s \in \mathbb{C}$, which is holomorphic at $s = 0$. Define the determinant $\det' \Delta^i$ by the formula

$$\log \det' \Delta^i = \frac{d}{ds} \zeta_i^{RS}(0).$$

Definition 3.3. The Ray-Singer torsion ρ^{RS} is defined by the formula (cf. [BZ1, Def. 2.2])

$$\log \rho^{RS} = \frac{1}{2} \sum_{i=0}^n (-1)^i i \log \det' \Delta^i.$$

The Ray-Singer metric $\|\cdot\|_{\det H^\bullet(M, F)}^{RS}$ on the line $\det H^\bullet(M, F)$ is the product

$$\|\cdot\|_{\det H^\bullet(M, F)}^{RS} = |\cdot|_{\det H^\bullet(M, F)}^{RS} \cdot \rho^{RS}.$$

Ray and Singer [RS, Th. 2.1] proved that the metric $\|\cdot\|_{\det H^\bullet(M, F)}^{RS}$ is a topological invariant, i.e., does not depend on the metrics g^{TM} or g^F .

4. THE WITTEN DEFORMATION.

4.1. A simplifying assumption. Recall that g^{TM} denotes the Riemannian metric on M . Following [BFK2] we give the following

Definition 4.2. *The pair (g^{TM}, f) is a generalized triangulation of M , if f is a Morse function $f : M \rightarrow \mathbb{R}$ satisfying the Thom-Smale transversality condition (cf. Sect. 2.4) and in a neighborhood of every critical point x of f one can introduce local coordinates (y_1, \dots, y_n) such that*

$$f(y) = f(x) - \frac{1}{2}(y_1^2 + \dots + y_k^2) + \frac{1}{2}(y_{k+1}^2 + \dots + y_n^2),$$

and the metric g^{TM} is Euclidean in these coordinates.

Since both the Milnor and the Ray-Singer metrics are independent of the choice of f and g^{TM} , it is enough to prove Theorem 1.2 for the case when (g^{TM}, f) is a generalized triangulation, which we will henceforth assume.

4.3. The Witten deformation of the Laplacian. Set $d_t^F = e^{-tf} d^F e^{tf}$, $d_t^{F*} = e^{tf} d^F e^{-tf}$. Then d_t^{F*} is the formal adjoint of d_t^F with respect to the scalar product (3.1). The *Witten Laplacian* is the operator

$$\Delta_{f,t} = d_t^F d_t^{F*} + d_t^{F*} d_t^F.$$

We denote by $\Delta_{f,t}^i$ the restriction of $\Delta_{f,t}$ to $\Omega^i(M, F)$. Let $\rho^{RS}(f, t)$ be the torsion defined as in Sect. 3.2, but with replacing everywhere Δ by $\Delta_{f,t}$.

The following theorem is well known, cf. [Wi], [Sh2, Prop. 2.7], [BFKM, Prop. 5.2]

Theorem 4.4. *Suppose that the pair (g^{TM}, f) is a generalized triangulation.*

1. *There exist positive constants C', C'' , and $t_0 > 1/C''$, so that for $|t| \geq t_0$, we have $\text{spec}(\Delta_{f,t}) \subset [0, e^{-|t|C'}] \cup (C''|t|, \infty)$.*

2. *Let $E_{sm,t}^\bullet \subset \Omega^\bullet(M, F)$ denote the span of the eigenvectors of $\Delta_{f,t}$ with eigenvalues less than $e^{-|t|C'}$. Then $\dim E_{sm,t}^i = m_i \text{rk}(F)$, $i = 1, \dots, n$, where m_i is the number of the critical points of f with index i .*

Clearly, $E_{sm,t}^\bullet$ is a subcomplex of the complex $(\Omega^\bullet(M, F), d_t^F)$. Let $\rho_{sm}^{RS}(f, t)$ be the torsion of this subcomplex. Let $P_{sm,t}^i : \Omega^i(M, F) \rightarrow E_{sm,t}^i$ be the orthogonal projection and let $P_{la,t}^i = 1 - P_{sm,t}^i$. Set $\zeta_{la,i}^{RS}(s) = -\text{Tr} [(\Delta_{f,t}^i)^{-s} P_{la,t}^i]$ and

$$\begin{aligned} \log \det' [\Delta_{f,t}^i P_{la,t}^i] &= \frac{d}{ds} \zeta_{la,i}^{RS}(0); \\ \log \rho_{la}^{RS}(f, t) &= \frac{1}{2} \sum_{i=0}^n (-1)^i i \log \det' [\Delta_{f,t}^i P_{la,t}^i]. \end{aligned} \tag{4.1}$$

Clearly,

$$\rho^{RS}(f, t) = \rho_{la}^{RS}(f, t) \cdot \rho_{sm}^{RS}(f, t) \quad \text{for } |t| > t_0. \tag{4.2}$$

4.5. The Witten Laplacian and the Ray-Singer metric. For each $t \in \mathbb{R}$, consider the metric $g_t^F = e^{-2tf} g^F$. Let $|\cdot|_{\det H^\bullet(M,F),f,t}^{RS}$ be the L^2 -metric on $\det H^\bullet(M,F)$ associated to the metrics g_t^F and g^{TM} . The Laplacian $\tilde{\Delta}_{f,t}$ associated to the metrics g_t^F and g^{TM} is conjugate to $\Delta_{f,t}$. More precisely, we have, $\tilde{\Delta}_{f,t} = e^{tf} \Delta_{f,t} e^{-tf}$, cf. [BZ1, Prop. 5.4]. Hence, $\rho^{RS}(f,t)$ equals the Ray-Singer torsion associated to the metrics g_t^F and g^{TM} . Since the Ray-Singer metric is a topological invariant of F , it follows that

$$\|\cdot\|_{\det H^\bullet(M,F)}^{RS} = |\cdot|_{\det H^\bullet(M,F),f,t}^{RS} \cdot \rho^{RS}(f,t), \quad \text{for any } t \in \mathbb{R}. \quad (4.3)$$

Theorem 4.6. *Suppose that the pair (g^{TM}, f) is a generalized triangulation, cf. Def. 4.2. Then, as $t \rightarrow +\infty$, we have*

$$\log \frac{|\cdot|_{\det H^\bullet(M,F)}^{RS} \rho_{sm}^{RS}(f,t)}{\|\cdot\|_{\det H^\bullet(M,F)}^M} = -t \operatorname{rk}(F) \operatorname{Tr}_s^B[f] + \frac{1}{2} \tilde{\chi}'(F) \log\left(\frac{t}{\pi}\right) + o(1), \quad (4.4)$$

where $\operatorname{Tr}_s^B[f] = \sum_{x \in B} (-1)^{\operatorname{ind}(x)} f(x)$ and $\tilde{\chi}'(F) = \operatorname{rk}(F) \sum_{x \in B} (-1)^{\operatorname{ind}(x)} \operatorname{ind}(x)$.

The theorem was first proven in [BZ1, Th. 7.6] using the difficult results of Helffer and Sjöstrand [HS]. A short and very elegant proof was found by Bismut and Zhang [BZ2, §6] (see also [BFKM]).

Recall that the number $R(M, F, f)$ was defined in (1.2). Using (4.2) and (4.3), we obtain the following corollary of Theorem 4.6:

Corollary 4.7. *There exists a constant $R = R(M, F, f)$, such that*

$$\log \rho_{la}^{RS}(f,t) = R(M, F, f) + t \operatorname{rk}(F) \operatorname{Tr}_s^B[f] - \frac{1}{2} \tilde{\chi}'(F) \log\left(\frac{t}{\pi}\right) + o(1), \quad t \rightarrow +\infty. \quad (4.5)$$

R is independent of f and Theorem 1.2 is equivalent to the equality $R = 0$.

Thus the proof of Theorem 1.2 is reduced to the study of the asymptotic expansion of $\rho_{la}^{RS}(f,t)$ as $t \rightarrow \infty$.

5. THE COMPARISON THEOREM

5.1. Let M, \tilde{M} be Riemannian manifolds of the same odd dimension n . Let F, \tilde{F} be flat vector bundles over M and \tilde{M} respectively, equipped with Hermitian metrics, such that the induced metrics on $\det F$ and $\det \tilde{F}$ are flat. We assume that $\dim F = \dim \tilde{F}$.

Definition 5.2. *The Morse functions $f : M \rightarrow \mathbb{R}$ and $\tilde{f} : \tilde{M} \rightarrow \mathbb{R}$ have the same critical points structure if there exist open neighborhoods $U \subset M$, $\tilde{U} \subset \tilde{M}$ of the sets of critical points of f, \tilde{f} respectively, and an isometry $\phi : U \rightarrow \tilde{U}$, such that $f = \tilde{f} \circ \phi$.*

Definition 5.3. *We say that a function $l(t)$ has a nice asymptotic expansion as $t \rightarrow \pm\infty$ if*

$$l(t) = \sum_{j=0}^n a_j (t/|t|) t^j + \sum_{k=0}^n b_k (t/|t|) t^k \log |t| + o(1),$$

and the coefficient a_0 (the free term) satisfy the equality $a_0(1) + a_0(-1) = 0$.

The main result of this section is the following

Theorem 5.4. *Let $f : M \rightarrow \mathbb{R}$, $\tilde{f} : \tilde{M} \rightarrow \mathbb{R}$ be Morse functions with the same critical points structure and let U, \tilde{U} be as in Def. 5.2. Then the difference $\log \rho_{la}^{RS}(f, t) - \log \rho_{la}^{RS}(\tilde{f}, t)$ has a nice asymptotic expansion.*

The rest of this section is devoted to the proof of Theorem 5.4.

5.5. Determinant of an operator almost elliptic with parameter. It is more convenient to work in a slightly more general situation. Suppose E is a Hermitian vector bundle over a compact Riemannian manifold M of dimension n . Consider the operator

$$H_t := A + tB + t^2V : C^\infty(M, E) \rightarrow C^\infty(M, E), \quad t \in \mathbb{R}, \quad (5.1)$$

where $A : C^\infty(M, E) \rightarrow C^\infty(M, E)$ is a second order self-adjoint elliptic differential operator with positive definite leading symbol, $B = B(x), V = V(x) : E \rightarrow E$ are self-adjoint bundle maps and $V(x) \geq 0$ for all $x \in M$. Suppose that there exist constants $t_0, c_1, c_2 > 0$ such that for all $|t| > t_0$, there are finitely many eigenvalues of H_t , which are smaller than $e^{-c_1|t|}$, while all the other eigenvalues of H_t are larger than $c_2|t|$. Let P_t denote the orthogonal projection onto the span of the eigensections of H_t with eigenvalues greater than 1.

Note that $\text{rk}(\text{Id} - P_t)$ is equal to the number of eigenvalues of H_t (counting multiplicities) which are smaller than 1. Hence, the function $t \mapsto \text{rk}(\text{Id} - P_t)$ is locally constant for $|t| > \max\{t_0, 1/c_2\}$. Set

$$m_\pm := \text{rk}(\text{Id} - P_t), \quad \pm t > \max\{t_0, 1/c_2\}. \quad (5.2)$$

Assume, in addition, that there exist constants $k > n/2$ and $C > 0$ such that

$$\text{Tr} [(H_t^k + \varepsilon)^{-1} P_t] < C, \quad \text{for all } \varepsilon > 0, |t| \gg 0. \quad (5.3)$$

Note that when $H_t = \Delta_{t,f}$ this assumption is satisfied for every $k > n$ by [BFK2, Lemma 3.3].

We are interested in the behaviour of the function $l(t) = \log \det' H_t P_t$, as $t \rightarrow \pm\infty$. Note that, if $V(x) > 0$ for all $x \in M$, then H_t is an *elliptic operator with parameter*, cf. [Sh1, Ch. 2], [BFK1, Appendix]. Then $l(t)$ has a nice asymptotic expansion as $t \rightarrow \infty$ with computable coefficients, cf. [BFK1, Appendix]. If $V(x)$ is not strictly positive for some $x \in M$, this asymptotic expansion does not hold any more. However, the following result is true: Let \tilde{E} be a Hermitian vector bundle over another compact Riemannian manifold \tilde{M} . We assume that the rank of E is equal to the rank of \tilde{E} . Let

$$\tilde{H}_t = \tilde{A} + t\tilde{B} + t^2\tilde{V} : C^\infty(\tilde{M}, \tilde{E}) \rightarrow C^\infty(\tilde{M}, \tilde{E})$$

be as above. Then 1 is not an eigenvalue of \tilde{H}_t for $|t| \gg 0$. Let \tilde{P}_t be the orthogonal projector onto the span of eigensections of \tilde{H}_t with eigenvalues greater than 1.

Theorem 5.6. *Suppose there exist open sets $U \subset M$ and $\tilde{U} \subset \tilde{M}$ such that $V(x) > 0$ for all $x \in M \setminus U$ and $\tilde{V}(x) > 0$ for all $x \in \tilde{M} \setminus \tilde{U}$. Let $\phi : U \rightarrow \tilde{U}$ be a diffeomorphism which preserves the Riemannian metric. Assume that $\psi : \phi^* \tilde{E}|_{\tilde{U}} \rightarrow E|_U$ is an isometry, which identifies the restrictions of H_t to U and of \tilde{H}_t to \tilde{U} . Then the function $\log \det' H_t P_t - \log \det' \tilde{H}_t \tilde{P}_t$ has a nice asymptotic expansion.*

Clearly, Theorem 5.4 is an immediate consequence of Theorem 5.6. We pass now to the proof of Theorem 5.6. First we establish the following

Proposition 5.7. *For every $k > n/2$, the following equality holds*

$$k \log \det' H_t P_t = \log \det' H_t^k P_t = \log \det' [(H_t^k + t^{2k}) P_t] - t^{2k} \int_0^1 \operatorname{Tr} [(H_t^k + \tau t^{2k})^{-1} P_t] d\tau. \quad (5.4)$$

Proof. For $k > n/2$ the operator $[(H_t^k + \tau t^{2k}) P_t]^{-1}$ is of trace class. Hence

$$\begin{aligned} \frac{d}{d\tau} \log \det' [(H_t^k + \tau t^{2k}) P_t] \\ = \operatorname{Tr} [(H_t^k + \tau t^{2k})^{-1} \frac{d}{d\tau} (H_t^k + \tau t^{2k}) P_t] = t^{2k} \operatorname{Tr} [(H_t^k + \tau t^{2k})^{-1} P_t]. \end{aligned}$$

Integrating this equality, we obtain (5.4). \square

From now on we fix $k > n/2$ such that (5.3) is satisfied.

Lemma 5.8. *As $t \rightarrow \infty$ we have*

$$\int_0^1 \operatorname{Tr} [(H_t^k + \tau t^{2k})^{-1} P_t] d\tau = \int_0^1 \operatorname{Tr} [(H_t^k + \tau t^{2k} + |t|^{-k})^{-1} P_t] d\tau + o(t^{-2k}). \quad (5.5)$$

Proof. By the definition of P_t we have $\| (H_t^k + \tau t^{2k})^{-1} P_t \| \leq ((c_2 t)^k + \tau t^{2k})^{-1}$. Using this inequality and (5.3) we have

$$\begin{aligned} \int_0^1 \operatorname{Tr} [(H_t^k + \tau t^{2k})^{-1} P_t] d\tau - \int_0^1 \operatorname{Tr} [(H_t^k + \tau t^{2k} + |t|^{-k})^{-1} P_t] d\tau \\ = |t|^{-k} \int_0^1 \operatorname{Tr} [(H_t^k + \tau t^{2k})^{-1} (H_t^k + \tau t^{2k} + |t|^{-k})^{-1} P_t] d\tau \\ \leq |t|^{-k} \int_0^1 \| (H_t^k + \tau t^{2k})^{-1} P_t \| \operatorname{Tr} [(H_t^k + \tau t^{2k} + |t|^{-k})^{-1} P_t] d\tau \\ \leq \frac{|t|^{-k}}{C} \int_0^1 \frac{1}{(c_2 t)^k + \tau t^{2k}} d\tau = \frac{|t|^{-3k}}{C} \log (1 + t^k/c_2^k) = o(t^{-2k}). \end{aligned}$$

\square

Recall that the numbers m_{\pm} were defined in (5.2). Clearly

$$\begin{aligned} \int_0^1 \operatorname{Tr} [(H_t^k + \tau t^{2k} + |t|^{-k})^{-1} P_t] d\tau \\ = \int_0^1 \operatorname{Tr} (H_t^k + \tau t^{2k} + |t|^{-k})^{-1} d\tau - 3km_{\pm} t^{-2k} \log |t| + o(t^{-2k}), \quad (5.6) \\ \log \det' [(H_t^k + t^{2k}) P_t] = \log \det' (H_t^k + t^{2k}) - 2km_{\pm} \log |t| + o(1), \end{aligned}$$

as $t \rightarrow \pm\infty$.

It is shown in the Appendix to [BFK2] that $\log \det'(H_t^k + t^{2k})$ has a nice asymptotic expansion. Hence, Theorem 5.6 follows from (5.4), (5.5), (5.6) and the following

Proposition 5.9. *Under the assumptions of Theorem 5.6 the function*

$$t^{2k} \int_0^1 \left[\operatorname{Tr}(H_t^k + \tau t^{2k} + |t|^{-k})^{-1} - \operatorname{Tr}(\tilde{H}_t^k + \tau t^{2k} + |t|^{-k})^{-1} \right] d\tau \quad (5.7)$$

has a nice asymptotic expansion.

The rest of this section is occupied with the proof of Prop. 5.9.

5.10. Notations. Let $W \subset M$ be an open set whose closure $\overline{W} \subset U$ and such that $V(x) > 0$ for all $x \notin W$. Fix $\varepsilon > 0$ such that $V(x) > \varepsilon$ for all $x \notin W$. We can and we will assume that $\varepsilon < 1$. Let $v : M \rightarrow [0, \varepsilon]$ be a smooth function such that $\operatorname{supp} v \subset U$ and $v|_W \equiv \varepsilon$. Set

$$A_{t,\tau} := H_t^k + \tau t^{2k} + |t|^{-k}, \quad A_{t,\tau,v} := H_t^k + \tau t^{2k} + |t|^{-k} + v^2 t^{2k}. \quad (5.8)$$

To simplify the notation we will identify U and \tilde{U} via the diffeomorphism $\phi : U \rightarrow \tilde{U}$. In particular, we will consider v as a function on \tilde{M} . We define operators $\tilde{A}_{t,\tau}$ and $\tilde{A}_{t,\tau,v}$ as in (5.8) but using \tilde{H}_t instead of H_t .

Lemma 5.11. *Let $K_{\tau,v}(t, x, y)$ denote the Schwartz kernel of the operator $A_{t,\tau,v}^{-1}$. Then for each $N \in \mathbb{N}$*

$$K_{\tau,v}(t, x, x) = \sum_{j=0}^N \alpha_j(\tau, t/|t|, x) t^{n-j-2k} + r_{N,\tau}(t, x), \quad (5.9)$$

where $t^N r_{N,\tau}(t, x) \rightarrow 0$ as $t \rightarrow \pm\infty$ uniformly in $\tau \in [0, 1]$, $x \in M$. The coefficients $\alpha_j(\tau, \pm 1, x)$ depend continuously on $\tau \in [0, 1]$ and can be expressed in terms of the full symbol of H_t and a finite number of its derivatives. If $j = 2i$ is even, then $\alpha_j(\tau, 1, x) + \alpha_j(\tau, -1, x) = 0$.

Proof. Clearly, $A_{t,\tau,v} = H_t^k + \tau t^{2k} + |t|^{-k} + v^2 t^{2k}$ is an operator elliptic with parameter, cf. [Sh1, Ch. 2], [BFK1, Appendix]. The lemma is a consequence of the standard construction of the parametrix of an operator elliptic with parameter. It follows immediately, for example, from Lemma A.8 in [BFK1]. \square

Since $\operatorname{Tr} A_{t,\tau,v}^{-1} = \int_M K_{\tau,v}(t, x, x) dx$ we obtain the following

Corollary 5.12. *The function $t^{2k} \int_0^1 \operatorname{Tr} A_{t,\tau,v}^{-1} d\tau$ has a nice asymptotic expansion.*

Prop. 5.9 (and, hence, Theorems 5.6 and 5.4) follows now from the following

Lemma 5.13. *Under the assumptions of Theorem 5.6 we have*

$$\operatorname{Tr} [A_{t,\tau}^{-1} - A_{t,\tau,v}^{-1}] - \operatorname{Tr} [\tilde{A}_{t,\tau}^{-1} - \tilde{A}_{t,\tau,v}^{-1}] = o(t^{-2k}) \quad (5.10)$$

as $t \rightarrow \infty$ uniformly in $\tau \in [0, 1]$.

Proof. We have

$$A_{t,\tau}^{-1} - A_{t,\tau,v}^{-1} = A_{t,\tau}^{-1} v^2 t^{2k} A_{t,\tau,v}^{-1} = A_{t,\tau,v}^{-1} v^2 t^{2k} A_{t,\tau}^{-1}. \quad (5.11)$$

Hence

$$\begin{aligned} \operatorname{Tr} \left[A_{t,\tau}^{-1} - A_{t,\tau,v}^{-1} \right] &= \operatorname{Tr} \left[A_{t,\tau}^{-1} v^2 t^{2k} A_{t,\tau,v}^{-1} \right] = t^{2k} \operatorname{Tr} \left[v A_{t,\tau,v}^{-1} A_{t,\tau}^{-1} v \right] \\ &= t^{2k} \operatorname{Tr} \left[v A_{t,\tau,v}^{-2} v \right] + t^{4k} \operatorname{Tr} \left[v A_{t,\tau,v}^{-2} v^2 A_{t,\tau}^{-1} v \right]. \end{aligned} \quad (5.12)$$

Similar equality is true for $\operatorname{Tr} \left[\tilde{A}_{t,\tau}^{-1} - \tilde{A}_{t,\tau,v}^{-1} \right]$.

Let $\tilde{K}_{\tau,v}(t, x, y)$ denote the Schwartz kernel of the operator $\tilde{A}_{t,\tau,v}^{-1}$. By Lemma 5.11, for all $x \in \operatorname{supp} v \subset U$ and all $N \in \mathbb{N}$ we have $K_{\tau,v}(t, x, x) - \tilde{K}_{\tau,v}(t, x, x) = o(t^{-N})$ as $t \rightarrow \infty$ uniformly in $\tau \in [0, 1]$. Hence,

$$\operatorname{Tr} \left[v A_{t,\tau,v}^{-2} v \right] - \operatorname{Tr} \left[v \tilde{A}_{t,\tau,v}^{-2} v \right] = \int_M v \left(K_{\tau,v}(t, x, x) - \tilde{K}_{\tau,v}(t, x, x) \right) v dx = o(t^{-N}), \quad (5.13)$$

as $t \rightarrow \infty$ uniformly in $\tau \in [0, 1]$.

Let $I_{t,\tau,v}$ denote the left hand side of (5.10). From (5.12) and (5.13) we conclude that

$$I_{t,\tau,v} := t^{4k} \operatorname{Tr} \left[v A_{t,\tau,v}^{-2} v^2 A_{t,\tau}^{-1} v \right] - t^{4k} \operatorname{Tr} \left[v \tilde{A}_{t,\tau,v}^{-2} v^2 \tilde{A}_{t,\tau}^{-1} v \right] + o(t^{2k-N}) \quad (5.14)$$

as $t \rightarrow \infty$ uniformly in $\tau \in [0, 1]$.

Using the isometry $\psi : \phi^* \tilde{E}|_{\tilde{U}} \rightarrow E|_U$ we can view $v \tilde{A}_{t,\tau,v}^{-2} v$ and $v \tilde{A}_{t,\tau}^{-2} v$ as operators acting on the space of sections of the bundle E . Then

$$\begin{aligned} & \left| \operatorname{Tr} \left[v A_{t,\tau,v}^{-2} v^2 A_{t,\tau}^{-1} v \right] - \operatorname{Tr} \left[v \tilde{A}_{t,\tau,v}^{-2} v^2 \tilde{A}_{t,\tau}^{-1} v \right] \right| \\ & \leq \left| \operatorname{Tr} \left[(v A_{t,\tau,v}^{-2} v - v \tilde{A}_{t,\tau,v}^{-2} v) v A_{t,\tau}^{-1} v \right] \right| + \left| \operatorname{Tr} \left[v \tilde{A}_{t,\tau,v}^{-2} v (v A_{t,\tau}^{-1} v - v \tilde{A}_{t,\tau}^{-1} v) \right] \right| \\ & \leq \|v A_{t,\tau}^{-1} v\| \cdot \left| \operatorname{Tr} \left[v A_{t,\tau,v}^{-2} v - v \tilde{A}_{t,\tau,v}^{-2} v \right] \right| + \|v \tilde{A}_{t,\tau,v}^{-2} v\| \cdot \left| \operatorname{Tr} \left[v A_{t,\tau}^{-1} v - v \tilde{A}_{t,\tau}^{-1} v \right] \right|. \end{aligned} \quad (5.15)$$

Fix $N > 7k$. Then, using (5.13), (5.14), (5.15) and the obvious estimates

$$\|v A_{t,\tau}^{-1} v\| \leq |t|^k, \quad \|v \tilde{A}_{t,\tau,v}^{-2} v\| \leq \varepsilon^{-2} t^{-4k},$$

we conclude

$$|I_{t,\tau,v}| \leq \varepsilon^{-2} \left| \operatorname{Tr} \left[v A_{t,\tau}^{-1} v - v \tilde{A}_{t,\tau}^{-1} v \right] \right| + o(t^{-2k}). \quad (5.16)$$

Applying again (5.11), we get

$$\begin{aligned} v A_{t,\tau}^{-1} v - v \tilde{A}_{t,\tau}^{-1} v &= t^{2k} \left(v A_{t,\tau,v}^{-1} v^2 A_{t,\tau}^{-1} v - v \tilde{A}_{t,\tau,v}^{-1} v^2 \tilde{A}_{t,\tau}^{-1} v \right) + \left(v A_{t,\tau,v}^{-1} v - v \tilde{A}_{t,\tau,v}^{-1} v \right) \\ &= t^{2k} v A_{t,\tau,v}^{-1} v \left(v A_{t,\tau}^{-1} v - v \tilde{A}_{t,\tau}^{-1} v \right) + t^{2k} \left(v A_{t,\tau,v}^{-1} v - v \tilde{A}_{t,\tau,v}^{-1} v \right) v \tilde{A}_{t,\tau}^{-1} v \\ & \quad + \left(v A_{t,\tau,v}^{-1} v - v \tilde{A}_{t,\tau,v}^{-1} v \right) \end{aligned} \quad (5.17)$$

As in (5.13), Lemma 5.11 implies that for all $N \in \mathbb{N}$ the traces of the second and the third summands in the right hand side of (5.17) behave as $o(t^{-N})$ when $t \rightarrow \infty$ uniformly in $\tau \in [0, 1]$. Thus

$$\left| \operatorname{Tr} \left[v A_{t,\tau}^{-1} v - v \widetilde{A}_{t,\tau}^{-1} v \right] \right| \leq t^{2k} \|v A_{t,\tau,v}^{-1} v\| \cdot \left| \operatorname{Tr} \left[v A_{t,\tau}^{-1} v - v \widetilde{A}_{t,\tau}^{-1} v \right] \right| + o(t^{-N}). \quad (5.18)$$

Clearly, $t^{2k} \|v A_{t,\tau,v}^{-1} v\| \leq \frac{\varepsilon^2 t^{2k}}{\varepsilon^2 t^{2k} + |t|^{-k}} \leq 1 - \frac{\varepsilon^{-2} |t|^{-3k}}{2}$. Hence, from (5.18) we conclude

$$\left| \operatorname{Tr} \left[v A_{t,\tau}^{-1} v - v \widetilde{A}_{t,\tau}^{-1} v \right] \right| \leq o(t^{3k-N}). \quad (5.19)$$

Taking $N > 5k$ we obtain from (5.16) and (5.19) that $I_{t,\tau,v} = o(t^{-2k})$ uniformly in $\tau \in [0, 1]$. \square

6. PROOF OF CHEEGER-MÜLLER THEOREM

Recall from Corollary 4.7, that to prove Theorem 1.2 it is enough to show that $R(M, F, f) = 0$. We will use the notation of Sect. 4. Clearly $\Delta_{f,t} = \Delta_{-f,-t}$. Hence,

$$\rho_{la}^{RS}(f, -t) = \rho_{la}^{RS}(-f, t). \quad (6.1)$$

Recall that the number $R(M, F, f)$ is defined in (1.2). Let $\widetilde{M}, \widetilde{F}, \widetilde{f}$ be as in Sect. 5.1. It follows from Corollary 4.7 that $R(M, F, f) - R(\widetilde{M}, \widetilde{F}, \widetilde{f})$ is equal to the free term of the asymptotic expansion of $\log \rho_{la}^{RS}(f, t) - \log \rho_{la}^{RS}(\widetilde{f}, t)$. Hence, from Theorem 5.4 and (6.1), we conclude

$$\left[R(M, F, f) - R(\widetilde{M}, \widetilde{F}, \widetilde{f}) \right] + \left[R(M, F, -f) - R(\widetilde{M}, \widetilde{F}, -\widetilde{f}) \right] = 0.$$

Since $R(M, F, f)$ is independent of f , cf. (1.2), we obtain

$$R(M, F, f) = R(\widetilde{M}, \widetilde{F}, \widetilde{f}). \quad (6.2)$$

Lemma 6.1. *Suppose N is a compact manifolds of even dimension. Let \overline{F} be the flat Hermitian vector bundle induced by F on the product $M \times N$. Fix a generalized triangulation (g^{TN}, f^N) on N , cf. Def. 4.2. Let \overline{f} be the Morse function on $M \times N$ defined by the formula $\overline{f}(x, y) = f(x) + f^N(y)$, where $x \in M, y \in N$. Then*

$$\log \rho_{la}^{RS}(\overline{f}, t) = \chi(N) \log \rho_{la}^{RS}(f, t), \quad (6.3)$$

where $\chi(N)$ is the Euler characteristic of N .

The proof is a verbatim repetition of the proof of Theorem 2.5 in [RS] and will be omitted (In [RS], the equality (6.3) is proven with ρ_{la}^{RS} replaced by the “full” Ray-Singer torsion ρ^{RS}). Using Corollary 4.7 and Lemma 6.1, we obtain

$$R(M \times N, \overline{F}, \overline{f}) = \chi(N) R(M, F, f). \quad (6.4)$$

Substituting in (6.4), $N = S^2$ and $N = S^1 \times S^1$ we obtain respectively

$$R(M \times S^2, \overline{F}, \overline{f}) = 2R(M, F, f); \quad R(M \times S^1 \times S^1, \overline{F}, \overline{f}) = 0. \quad (6.5)$$

Using the results of [Mi1, §5] (see also Lemma 4.2 of [BFK2]), we see that there exist generalized triangulations $(g^{M \times S^2}, f_1)$ on $M \times S^2$ and $(g^{M \times S^1 \times S^1}, f_2)$ on $M \times S^1 \times S^1$, such that the functions f_1 and f_2 have the same critical points structure, cf. Def. 5.2. Hence, by (6.2), we have

$$R(M \times S^2, \overline{F}, f_1) = R(M \times S^1 \times S^1, \overline{F}, f_2). \quad (6.6)$$

From (6.5), (6.6) and the fact that R is independent of the choice of the Morse function, we obtain $R(M, F, f) = 0$. \square

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