Let \( \{U_\alpha : \alpha \in A\} \) be an arbitrary open cover. The set \( \overline{G}_i - G_{i-1} \) is compact and contained in the open set \( G_{i+1} - G_{i-2} \). For each \( i \geq 3 \) choose a finite subcover of the open cover \( \{U_\alpha \cap (G_{i+1} - G_{i-2}) : \alpha \in A\} \) of \( \overline{G}_i - G_{i-1} \), and choose a finite subcover of the open cover \( \{U_\alpha \cap G_\alpha : \alpha \in A\} \) of the compact set \( \overline{G}_\alpha \). This collection of open sets is easily seen to be a countable, locally finite refinement of the open cover \( \{U_\alpha\} \), and consists of open sets with compact closures.

1.10 Lemma There exists a non-negative \( C^\infty \) function \( \varphi \) on \( \mathbb{R}^d \) which equals 1 on the closed cube \( C(1) \) and zero on the complement of the open cube \( C(2) \).

**Proof** We need only let \( \varphi \) be the product

\[
\varphi = (h \circ r_1) \cdots (h \circ r_d),
\]

where \( h \) is a non-negative \( C^\infty \) function on the real line which is 1 on \([-1,1]\) and zero outside of \((-2,2)\). To construct such an \( h \), we start with the function

\[
f(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}
\]

which is non-negative, \( C^\infty \), and positive for \( t > 0 \). Then the function

\[
g(t) = \frac{f(t)}{f(t) + f(1-t)}
\]

is non-negative, \( C^\infty \), and takes the value 1 for \( t \geq 1 \) and the value zero for \( t \leq 0 \). We obtain the desired function \( h \) by setting

\[
h(t) = g(t + 2)g(2 - t).
\]

1.11 Theorem (Existence of Partitions of Unity) Let \( M \) be a differentiable manifold and \( \{U_\alpha : \alpha \in A\} \) an open cover of \( M \). Then there exists a countable partition of unity \( \{\varphi_\alpha : \alpha = 1, 2, 3, \ldots\} \) subordinate to the cover \( \{U_\alpha\} \) with \( \operatorname{supp} \varphi_\alpha \) compact for each \( \alpha \). If one does not require compact supports, then there is a partition of unity \( \{\varphi_\alpha\} \) subordinate to the cover \( \{U_\alpha\} \) (that is, \( \operatorname{supp} \varphi_\alpha \subset U_\alpha \)) with at most countably many of the \( \varphi_\alpha \) not identically zero.

**Proof** Let the sequence \( \{G_i\} \) cover \( M \) as in 1.9(1), and set \( G_0 = \emptyset \). For \( p \in M \), let \( i_p \) be the largest integer such that \( p \in M - \overline{G}_{i_p} \). Choose an \( \alpha_p \) such that \( p \in U_{\alpha_p} \), and let \( (V, \tau) \) be a coordinate system centered at \( p \) such that \( V \subset U_{\alpha_p} \cap (G_{i_p+1} - G_{i_p}) \) and such that \( \tau(V) \) contains the closed cube \( C(2) \). Define

\[
\varphi_p = \begin{cases} \varphi \circ \tau & \text{on } V \\ 0 & \text{elsewhere} \end{cases}
\]

where \( \varphi \) is the function 1.10(1). Then \( \varphi_p \) is a \( C^\infty \) function on \( M \) which has the value 1 on some open neighborhood \( W_\varphi \) of \( p \), and has compact support lying in \( V \subset U_{\alpha_p} \cap (G_{i_p+1} - G_{i_p}) \). For each \( i \geq 1 \), choose a finite set of points \( p \) in \( M \) whose corresponding \( W_\varphi \) neighborhoods cover \( \overline{G}_i - G_{i-1} \). Order the corresponding \( \varphi_p \) functions in a sequence \( \varphi_{p_j} : j = 1, 2, 3, \ldots \). The supports of the \( \varphi_p \) form a locally finite family of subsets of \( M \). Thus the function

\[
\psi = \sum_{j=1}^{\infty} \varphi_{p_j}
\]

is a well-defined \( C^\infty \) function on \( M \), and moreover \( \psi(p) > 0 \) for each \( p \in M \). For each \( i = 1, 2, 3, \ldots \) define

\[
\varphi_i = \frac{\varphi}{\psi}.
\]

Then the functions \( \{\varphi_i : i = 1, 2, 3, \ldots\} \) form a partition of unity subordinate to the cover \( \{U_\alpha\} \) with \( \operatorname{supp} \varphi_\alpha \) compact for each \( \alpha \). If we let \( \varphi_\alpha \) be identically zero if no \( \varphi_i \) has support in \( U_\alpha \), and otherwise let \( \varphi_\alpha \) be the sum of the \( \varphi_i \) with support in \( U_\alpha \), then \( \{\varphi_\alpha\} \) is a partition of unity subordinate to the cover \( \{U_\alpha\} \) with at most countably many of the \( \varphi_\alpha \) not identically zero. To see that the support of \( \varphi_\alpha \) lies in \( U_\alpha \), observe that if \( A \) is a locally finite family of closed sets, then \( \bigcup_{\alpha \in A} A = \bigcup_{\alpha \in A} A \). Observe, however, that the support of \( \varphi_\alpha \) is not necessarily compact.

**Corollary** Let \( G \) be open in \( M \), and let \( A \) be closed in \( M \), with \( A \subset G \). Then there exists a \( C^\infty \) function \( \varphi : M \to \mathbb{R} \) such that

(a) \( 0 \leq \varphi(p) \leq 1 \) for all \( p \in M \).

(b) \( \varphi(p) = 1 \) if \( p \in A \).

(c) \( \operatorname{supp} \varphi \subset G \).

**Proof** There is a partition of unity \( \{\varphi, \psi\} \) subordinate to the cover \( \{G, M - A\} \) of \( M \) with \( \operatorname{supp} \varphi \subset G \) and \( \operatorname{supp} \psi \subset M - A \). Then \( \varphi \) is the desired function.

**TANGENT VECTORS AND DIFFERENTIALS**

1.12 A vector \( v \) with components \( v_1, \ldots, v_d \) at a point \( p \) in Euclidean space \( \mathbb{R}^d \) can be thought of as an operator on differentiable functions. Specifically, if \( f \) is differentiable on a neighborhood of \( p \), then \( v \) assigns to \( f \) the real number \( v(f) \) which is the directional derivative of \( f \) in the direction \( v \) at \( p \). That is,

\[
v(f) = v_1 \left. \frac{\partial f}{\partial r_1} \right|_p + \cdots + v_d \left. \frac{\partial f}{\partial r_d} \right|_p.
\]
This operation of the vector \( v \) on differentiable functions satisfies two important properties,

\[
\begin{align*}
  v(f + \lambda g) &= v(f) + \lambda v(g), \\
  v(f \cdot g) &= (f) v(g) + (g) v(f),
\end{align*}
\]

whenever \( f \) and \( g \) are differentiable near \( p \), and \( \lambda \) is a real number. The first property says that \( v \) acts linearly on functions, and the second says that \( v \) is a \textit{derivation}. This motivates our definition of tangent vectors on manifolds. They will be directional derivatives, that is, linear derivations on functions. The operation of taking derivatives depends only on local properties of functions, properties in arbitrarily small neighborhoods of the point at which the derivative is being taken. In order to express most conveniently this dependence of the derivative on the local nature of functions, we introduce the notion of germs of functions.

1.13 Definitions Let \( m \in M \). Functions \( f \) and \( g \) defined on open sets containing \( m \) are said to have the same germ at \( m \) if they agree on some neighborhood of \( m \). This introduces an equivalence relation on the \( C^\infty \) functions defined on neighborhoods of \( m \), two functions being equivalent if and only if they have the same germ. The equivalence classes are called \textit{germs}, and we denote the set of germs at \( m \) by \( F_m \). If \( f \) is a \( C^\infty \) function on a neighborhood of \( m \), then \( f \) will denote its germ. The operations of addition, scalar multiplication, and multiplication of functions induce on \( F_m \) the structure of an algebra over \( \mathbb{R} \). A germ \( f \) has a well-defined value \( f(m) \) at \( m \), namely, the value at \( m \) of any representative of the germ. Let \( F_m \subset F_m \) be the set of germs which vanish at \( m \). Then \( F_m \) is an ideal in \( F_m \), and we let \( F_m^k \) denote its \( k \)-th power. \( F_m^k \) is the ideal of \( F_m \) consisting of all finite linear combinations of \( k \)-fold products of elements of \( F_m \). These form a descending sequence of ideals \( F_m \supset F_m^2 \supset F_m^3 \supset \ldots \).

1.14 Definition A tangent vector \( v \) at the point \( m \in M \) is a linear derivation of the algebra \( F_m \). That is, for all \( f, g \in F_m \) and \( \lambda \in \mathbb{R} \),

\[
\begin{align*}
  v(f + \lambda g) &= v(f) + \lambda v(g), \\
  v(f \cdot g) &= (f) v(g) + (g) v(f).
\end{align*}
\]

\( M_m \) denotes the set of tangent vectors to \( M \) at \( m \) and is called the tangent \textit{space to} \( M \) at \( m \). Observe that if we define \( v + w \) by (1)

\[ (v + w)(f) = v(f) + w(f) \]

and \( \lambda v \) by (2)

\[ (\lambda v)(f) = \lambda v(f) \]

whenever \( v, w \in M_m \) and \( \lambda \in \mathbb{R} \), then \( v + w \) and \( \lambda v \) again are tangent vectors at \( m \). So in this way \( M_m \) becomes a real vector space. The fundamental property of the vector space \( M_m \), which we shall establish in 1.17, is that its dimension equals the dimension of \( M \). This definition of tangent vector is not suitable in the \( C^k \) case for \( 1 \leq k < \infty \). (We will discuss the \( C^k \) case further in 1.21.) We give this definition of tangent vector for several reasons. One reason is that it is intrinsic; that is, it does not depend on coordinate systems. Another reason is that it generalizes naturally to higher order tangent vectors, as we shall see in 1.26.

1.15 If \( c \) is the germ of a function with the constant value \( c \) on a neighborhood of \( m \), and if \( v \) is a tangent vector at \( m \), then \( v(c) = 0 \), for

\[ v(c) = v(c(1)) = 1v(1) + 1v(1) = 2v(1). \]

1.16 Lemma \( M_m \) is naturally isomorphic with \((F_m/F_m^2)^*\). (The symbol \(^*\) denotes dual vector space.)

Proof If \( v \in M_m \), then \( v \) is a linear function on \( F_m \) vanishing on \( F_m^2 \) because of the derivation property. Conversely, if \( f \in (F_m/F_m^2)^* \), we define a tangent vector \( v_f \) at \( m \) by setting \( v_f(f) = \langle f - f(m) \rangle \) for \( f \in F_m \).

(Here \( f(m) \) denotes the germ of the function with the constant value \( f(m) \), and \( \langle \cdot \rangle \) is used to denote cosets in \( F_m/F_m^2 \).) Linearity of \( v_f \) on \( F_m \) is clear. It is a derivation since

\[
\begin{align*}
  v_f(f \cdot g) &= \langle f \cdot g - f(m)(g)(m) \rangle \\
  &= \langle (f - f(m))(g - g(m)) + f(m)(g - g(m)) \rangle \\
  &= \langle (f - f(m))(g - g(m)) \rangle + f(m)\langle g - g(m) \rangle \\
  &= f(m)v_f(g) + g(m)v_f(f).
\end{align*}
\]

Thus we obtain mappings of \( M_m \) into \((F_m/F_m^2)^*\), and vice versa. It is easily checked that these are inverses of each other and thus are isomorphisms.

1.17 Theorem \( \dim (F_m/F_m^2) = \dim M \).

The proof is based on the following calculus lemma [31].

Lemma If \( g \) is of class \( C^k \) \((k \geq 2)\) on a convex open set \( U \) about \( p \) in \( \mathbb{R}^d \), then for each \( q \in U \),

\[
\begin{align*}
  (g(q) &= g(p) + \sum \frac{\partial g}{\partial r_i} \bigg|_{p} (r_i(q) - r_i(p)) \\
  &+ \sum \left( r_i(q) - r_i(p) \right) (r_j(q) - r_j(p)) \int_{0}^{1} (1 - t) \frac{\partial^2 g}{\partial r_i \partial r_j} \bigg|_{(t p + (1 - t) q)} dt.
\end{align*}
\]

In particular, if \( g \in C^\infty \), then the second summation in (1) determines an element of \( F_p^2 \) since the integral as a function of \( q \) is of class \( C^\infty \).
Proof of 1.17  Let \((U, \varphi)\) be a coordinate system about \(m\) with coordinate functions \(x_1, \ldots, x_d\) \((d = \dim M)\). Let \(f \in F_m\). Apply (1) to \(f \circ \varphi^{-1}\), and compose with \(\varphi\) to obtain

\[
f = \sum_{i=1}^d \frac{\partial (f \circ \varphi^{-1})}{\partial r_i}\bigg|_{\varphi(m)} (x_i - x_i(m)) + \sum_{i,j} (x_j - x_i(m))(x_j - x_i(m)) h\]
on a neighborhood of \(m\), where \(h \in C^\infty\). Thus

\[
f \equiv \sum_{i=1}^d \frac{\partial (f \circ \varphi^{-1})}{\partial r_i}\bigg|_{\varphi(m)} (\tilde{x}_i - x_i(m)) \mod F_m^2.
\]

Hence \(\{(x_i - x_i(m)): i = 1, \ldots, d\}\) spans \(F_m/(F_m)^2\). Consequently \(\dim F_m/(F_m)^2 \leq d\). We claim that these elements are linearly independent. For suppose that

\[
\sum_{i=1}^d a_i (x_i - x_i(m)) \in F_m^2.
\]

Now,

\[
\sum_{i=1}^d a_i (x_i - x_i(m)) \circ \varphi^{-1} = \sum_{i=1}^d a_i (r_i - r_i(\varphi(m))).
\]

Thus

\[
\sum_{i=1}^d a_i (r_i - r_i(\varphi(m))) \in F_m^2.
\]

But this implies that

\[
\frac{\partial}{\partial r_i}\bigg|_{\varphi(m)} \left( \sum_{i=1}^d a_i (r_i - r_i(\varphi(m))) \right) = 0
\]

for \(j = 1, \ldots, d\), which implies that the \(a_i\) must all be zero.

Corollary  \(\dim M_m = \dim M\).

1.18 In practice we will treat tangent vectors as operating on functions rather than on their germs. If \(f\) is a differentiable function defined on a neighborhood of \(m\), and \(v \in M_m\), we define

\[
v(f) = v(f).
\]

Thus \(v(f) = v(g)\) whenever \(f\) and \(g\) agree on a neighborhood of \(m\), and clearly

\[
v(f + \lambda g) = v(f) + \lambda v(g) \quad (\lambda \in \mathbb{R}),
\]

\[
v(f \cdot g) = v(f) v(g) + g(m) v(f),
\]

where \(f + \lambda g\) and \(f \cdot g\) are defined on the intersection of the domains of definition of \(f\) and \(g\).

1.19 Definition  Let \((U, \varphi)\) be a coordinate system with coordinate functions \(x_1, \ldots, x_d\), and let \(m \in U\). For each \(i \in (1, \ldots, d)\), we define a tangent vector \((\partial/\partial x_i)|_m \in M_m\) by setting

\[
\left(\frac{\partial}{\partial x_i}\right)_m (f) = \frac{\partial (f \circ \varphi^{-1})}{\partial r_i}\bigg|_{\varphi(m)}.
\]

for each function \(f\) which is \(C^\infty\) on a neighborhood of \(m\). We interpret (1) as the directional derivative of \(f\) at \(m\) in the \(x_i\) coordinate direction. We also use the notation

\[
\frac{\partial f}{\partial x_i}|_m = \left(\frac{\partial}{\partial x_i}\right)_m (f).
\]

1.20 Remarks on 1.19

(a) Clearly \((\partial/\partial x_i)|_m (f)\) depends only on the germ of \(f\) at \(m\), and (a) and (b) of 1.14 are satisfied; so \((\partial/\partial x_i)|_m\) is a tangent vector at \(m\). Moreover, \(\{(\partial/\partial x_i)|_m: i = 1, \ldots, d\}\) is a basis of \(M_m\). Indeed, it is the basis of \(M_m\) dual to the basis \(\{(x_i - x_i(m)): i = 1, \ldots, d\}\) of \(F_m/F_m^2\) since

\[
\left(\frac{\partial}{\partial x_i}\right)_m (x_j - x_j(m)) = \delta_{ij}.
\]

(b) If \(v \in M_m\), then

\[
v = \sum_{i=1}^d v(x_i) \left(\frac{\partial}{\partial x_i}\right)_m.
\]

Simply check that both sides give the same results when applied to the functions \((x_j - x_j(m))\).

(c) Suppose that \((U, \varphi)\) and \((V, \psi)\) are coordinate systems about \(m\), with coordinate functions \(x_1, \ldots, x_d\) and \(y_1, \ldots, y_d\) respectively. Then it follows from remark (b) that

\[
\frac{\partial}{\partial y_j}|_m = \sum_{i=1}^d \frac{\partial x_i}{\partial y_j}|_m \left(\frac{\partial}{\partial x_i}\right)_m.
\]

Observe that \((\partial/\partial y_i)\) depends on \(\varphi\) and not only on \(x_i\). In particular, if \(x_i\) were equal to \(y_1\), it would not necessarily follow that \(\partial/\partial x_i\) equals \(\partial/\partial y_1\).
The Differential

Let \( \psi : M \to N \) be \( C^\infty \), and let \( m \in M \). The differential of \( \psi \) at \( m \) is the linear map

\[
\psi_\ast : M_m^\ast \to N_{\psi(m)}^\ast
\]

defined as follows. If \( v \in M_m^\ast \), then \( \psi_\ast(v) \) is to be a tangent vector at \( \psi(m) \), so we describe how it operates on functions. Let \( g \) be a \( C^\infty \) function on a neighborhood of \( \psi(m) \). Define \( \psi_\ast(v)(g) \) by setting

\[
\psi_\ast(v)(g) = v(g \circ \psi).
\]

It is easily checked that \( \psi_\ast(v) \) is a linear map of \( M_m^\ast \) into \( N_{\psi(m)}^\ast \). Strictly speaking, this map should be denoted \( \psi_\ast|_{M_m^\ast} \), or simply \( \psi_\ast \). However, we omit the subscript \( m \) when there is no possibility of confusion. The map \( \psi \) is called non-singular at \( m \) if \( \psi_\ast \) is non-singular, that is, if the kernel of \((1)\) consists of \( 0 \) alone. The dual map

\[
\psi_\ast : N_{\psi(m)}^\ast \to M_m^\ast
\]

is defined as usual by requiring that

\[
\psi_\ast(\omega)(v) = \omega(\psi_\ast(v))
\]

whenever \( \omega \in N_{\psi(m)}^\ast \) and \( v \in M_m^\ast \). In the special case of a \( C^\infty \) function \( f : M \to \mathbb{R} \), if \( v \in M_m^\ast \) and \( f(m) = r_0 \), then

\[
df(v) = v(f) \left. \frac{d}{dr} \right|_{r_0}.
\]
Now, if \( v \neq 0 \) is any element of \( M_m \), then \( v \) is the tangent vector to a smooth curve in \( M \). For one can simply choose a coordinate system \((U, \varphi)\), centered at \( m \), for which

\[
v = d \varphi^{-1} \left( \frac{\partial}{\partial r_1} \bigg|_m \right).
\]

Then \( v \) is the tangent vector at \( 0 \) to the curve \( t \mapsto \varphi^{-1}(t, 0, \ldots, 0) \). One should observe that many curves can have the same tangent vector, and that two smooth curves \( \sigma \) and \( \tau \) in \( M \) for which \( \sigma(t_0) = \tau(t_0) = m \) have the same tangent vector at \( t_0 \) if and only if

\[
\frac{d(f \circ \sigma)}{dr} \bigg|_{t_0} = \frac{d(f \circ \tau)}{dr} \bigg|_{t_0}
\]

for all functions \( f \) which are \( C^\infty \) on a neighborhood of \( m \).

If \( \sigma \) happens to be a curve in the Euclidean space \( \mathbb{R}^n \), then

\[
\frac{d\sigma_1}{dr} \bigg|_{t \sigma(t)} + \cdots + \frac{d\sigma_n}{dr} \bigg|_{t \sigma(t)}
\]

If we identify this tangent vector with the element

\[
\begin{pmatrix}
\frac{d\sigma_1}{dr} \\
\vdots \\
\frac{d\sigma_n}{dr}
\end{pmatrix}
\]

of \( \mathbb{R}^n \), then we have

\[
\sigma(t) = \lim_{h \to 0} \frac{\sigma(t + h) - \sigma(t)}{h}
\]

Thus with this identification our notion of tangent vector coincides, in this special case, with the geometric notion of a tangent to a curve in Euclidean space.

1.24 Theorem Let \( \psi \) be a \( C^\infty \) mapping of the connected manifold \( M \) into the manifold \( N \). Suppose that for each \( m \in M \), \( d\psi_m \equiv 0 \). Then \( \psi \) is a constant map.

Proof Let \( m \in \psi(M) \). \( \psi^{-1}(m) \) is closed. We need only show that it is open. For this, let \( m \in \psi^{-1}(m) \). Choose coordinate systems \((U, x_1, \ldots, x_d)\) and \((V, y_1, \ldots, y_c)\) about \( m \) and \( n \) respectively, so that \( \psi(U) \subset V \). Then on \( U \),

\[
0 = d\psi \left( \frac{\partial}{\partial x_j} \right) = \sum_{i=1}^c \frac{\partial(\psi \circ \varphi)}{\partial x_i} \frac{\partial}{\partial y_i} \quad (j = 1, \ldots, d),
\]

which implies that

\[
\frac{\partial(\psi \circ \varphi)}{\partial x_i} \equiv 0 \quad (i = 1, \ldots, c; j = 1, \ldots, d).
\]

Thus the functions \( y_i \circ \psi \) are constant on \( U \). This implies that \( \psi(U) = n \); hence \( \psi^{-1}(n) \) is open and consequently \( \psi^{-1}(n) = M \).

We shall now see that in a natural way the collection of all tangent vectors to a differentiable manifold itself forms a differentiable manifold called the tangent bundle. We have a similar dual object called the cotangent bundle formed from the linear functionals on the tangent spaces.

1.25 Tangent and Cotangent Bundles Let \( M \) be a \( C^\infty \) manifold with differentiable structure \( \mathcal{F} \). Let

\[
T(M) = \bigcup_{m \in M} M_m,
\]

(1)

\[
T^*(M) = \bigcup_{m \in M} M^*_m.
\]

There are natural projections:

\[
\pi: T(M) \to M, \quad \pi(v) = m \quad \text{if} \quad v \in M_m,
\]

\[
\pi^*: T^*(M) \to M, \quad \pi^*(\tau) = m \quad \text{if} \quad \tau \in M^*_m.
\]

Let \( (U, \varphi) \in \mathcal{F} \) with coordinate functions \( x_1, \ldots, x_d \). Define \( \tilde{\varphi}: \pi^{-1}(U) \to \mathbb{R}^d \) and \( \tilde{\varphi}^*: (\pi^*)^{-1}(U) \to \mathbb{R}^d \) by

\[
\tilde{\varphi}(v) = (x_1(\pi(v)), \ldots, x_d(\pi(v)), dx_1(v), \ldots, dx_d(v))
\]

\[
\tilde{\varphi}^*(\tau) = (x_1(\pi^*(\tau)), \ldots, x_d(\pi^*(\tau)), \tau \left( \frac{\partial}{\partial x_1} \right), \ldots, \tau \left( \frac{\partial}{\partial x_d} \right))
\]

for all \( v \in \pi^{-1}(U) \) and \( \tau \in (\pi^*)^{-1}(U) \). Note that \( \tilde{\varphi} \) and \( \tilde{\varphi}^* \) are both one-to-one maps onto open subsets of \( \mathbb{R}^d \). The following steps outline the construction of a topology and a differentiable structure on \( T(M) \). The construction for \( T^*(M) \) goes similarly. The proofs are left as exercises.

(a) If \( (U, \varphi) \) and \( (V, \psi) \in \mathcal{F} \), then \( \varphi \circ \psi^{-1} \) is \( C^\infty \).

(b) The collection \( \{ \varphi^{-1}(W) : W \text{ open in } \mathbb{R}^d, (U, \varphi) \in \mathcal{F} \} \) forms a basis for a topology on \( T(M) \) which makes \( T(M) \) into a \( 2d \)-dimensional, second countable, locally Euclidean space.

(c) Let \( \mathcal{F} \) be the maximal collection, with respect to 1.4(b), containing

\[
\{ \pi^{-1}(U), \tilde{\varphi} : (U, \varphi) \in \mathcal{F} \}.
\]

Then \( \mathcal{F} \) is a differentiable structure on \( T(M) \).

\( T(M) \) and \( T^*(M) \) with these differentiable structures are called respectively the tangent bundle and the cotangent bundle. It will sometimes be convenient to write the points of \( T(M) \) as pairs \((m, v)\) where \( m \in M \) and \( v \in M_m \) (and similarly for \( T^*(M) \)).
If \( \psi: M \to N \) is a \( C^\infty \) map, then the differential of \( \psi \) defines a mapping of the tangent bundles

\[
d\psi: T(M) \to T(N),
\]
where \( d\psi(m,v) = d\psi_m(v) \) whenever \( v \in TM \). It is easily checked that (4) is a \( C^\infty \) map.

### 1.26 Higher Order Tangent Vectors and Differentials

It is useful to look at \( M^k \) as \( (\mathbb{R}^\ell \otimes \mathbb{R}^k)^* \), for this point of view allows an immediate generalization to higher order tangent vectors. We digress for a moment to give these definitions.

Recall that \( \mathbb{R}^\ell \) is the algebra of germs of functions at \( m \). \( F_m \subset \mathbb{R}^\ell \) is the ideal of germs vanishing at \( m \), and \( F^k_m \) \( (k \) an integer \( \geq 1) \) is the ideal of \( \mathbb{R}^\ell \) consisting of all finite linear combinations of \( k \)-fold products of elements of \( F_m \).

The vector space \( F_m^k/ F^{k+1}_m \) is called the space of \( k \)th order differentials at \( m \), and we denote it by \( \mathbb{M}^k_m \). As before, \( f \) denotes the germ of \( f \) at \( m \), and \( \{ \} \) will denote cosets in \( F_m^k/ F^{k+1}_m \). Let \( f \) be a differentiable function on a neighborhood of \( m \). We define the \( k \)th order differential \( df \) of \( f \) at \( m \) by

\[
d^k f = \{ f - f(m) \}
\]

A \( k \)th order tangent vector at \( m \) is a real linear function on \( F_m \) vanishing on \( F^{k+1}_m \) and vanishing also on the set of germs of functions constant on a neighborhood of \( m \). The real linear space of \( k \)th order tangent vectors at \( m \) will be denoted by \( \mathbb{M}^k_m \). We have a natural identification of \( \mathbb{R}^\ell \) with \( \mathbb{M}^0_m \) since any \( k \)th order tangent vector restricted to \( F_m \) yields a linear function on \( F_m \) vanishing on \( F^{k+1}_m \), and hence yields an element of \( \mathbb{R}^\ell \); conversely an element of \( \mathbb{R}^\ell \) uniquely determines a linear function on \( F_m \) vanishing on \( F^{k+1}_m \), and this extends uniquely to a \( k \)th order tangent vector by requiring it to annihilate germs of constant functions.

We can tie up this notion of higher order tangent vector with the usual notion of higher order derivative in Euclidean space by looking at the forms that these tangent vectors and differentials take in a coordinate system. Let \( (U, \varphi) \) be a coordinate system about \( m \) with coordinate functions \( x_1, \ldots, x_k \) such that \( \varphi(U) \) is a convex open set in Euclidean space \( \mathbb{R}^d \). Let \( x = (x_1, \ldots, x_d) \) be a list of non-negative integers. In addition to our conventions of 1.1, we let

\[
(x - x(m))^a = (x_1 - x_1(m))^a_1 \cdots (x_d - x_d(m))^a_d.
\]

Let \( f \) be a \( C^\infty \) function on \( U \). Then it follows from the lemma of 1.17, that

\[
f = f(m) + \sum_{\{a\}} b_a (x - x(m))^a + \sum_{\{a\} = a+1} h_a (x - x(m))^a,
\]

where the \( h_a \) are \( C^\infty \) functions on \( U \) and where

\[
a_a = \frac{1}{a!} \frac{\partial^a (f \circ \varphi^{-1})}{\partial x^a} \bigg|_{\varphi(m)}.
\]

Hence

\[
d^k f = \sum_{\{a\} \leq k} a_a (x - x(m))^a.
\]

Thus the collection

\[
[(x - x(m))^a] : 1 \leq \{a\} \leq k
\]

spans \( \mathbb{M}^k_m \). The proof that these elements are linearly independent in \( \mathbb{M}^k_m \) is the obvious generalization of the proof for the case \( k = 2 \) which was treated in 1.17. Thus the collection (5) forms a basis of \( \mathbb{M}^k_m \). Consequently \( \mathbb{M}^k_m \) is \( \text{finite dimensional} \) with dimension equal to the binomial coefficient

\[
\sum_{i=0}^{k-j} \binom{d+j-1}{j}.
\]

As the dual space of \( \mathbb{M}^k_m \), \( \mathbb{M}^k_m \) is also finite dimensional with the same dimension. Since \( \mathbb{M}^k_m \) is identified with \( \mathbb{R}^\ell \), and these spaces are finite dimensional, we have a canonical isomorphism of \( \mathbb{M}^k_m \) with \( \mathbb{M}^k_m \), under which the element of \( df \in \mathbb{M}^k_m \), considered as an element of \( \mathbb{M}^k_m \), satisfies

\[
d^k f(v) = v(f).
\]

Let

\[
\frac{\partial^a f}{\partial x^a} \bigg|_{m} = \frac{\partial^a (f \circ \varphi^{-1})}{\partial x^a} \bigg|_{\varphi(m)}.
\]

Since the derivative is linear, and since the value of \( \frac{\partial^a f}{\partial x^a} \) at \( m \) depends only on the germ of \( f \) at \( m \) and vanishes if \( f \) is constant on a neighborhood of \( m \) or if \( f \) is an \( \{a\} + 1 \)-fold product of functions which vanish at \( m \), then \( \frac{\partial^a f}{\partial x^a} \) is an \( \{a\} \)th order tangent vector at \( m \). It follows that

\[
\left( \frac{1}{a!} \frac{\partial^a}{\partial x^a} \bigg|_{m} \right) : 1 \leq \{a\} \leq k
\]

is the basis of \( \mathbb{M}^k_m \) dual to the basis (5) of \( \mathbb{M}^k_m \). If \( v \) is a \( k \)th order tangent vector at \( m \), then

\[
v = \sum_{\{a\} = 1}^{k} b_a \frac{\partial^a}{\partial x^a} \bigg|_{m},
\]

where

\[
b_a = \frac{1}{a!} v((x - x(m))^a).
\]

In terms of the basis (8), equation (3) becomes

\[
a_a = \frac{1}{a!} \frac{\partial f}{\partial x^a} \bigg|_{m}.
\]
As in the case of first order tangent vectors, we customarily think of
tangent vectors as operating on the functions themselves rather than their
germ; indeed, we define
\begin{equation}
\nu(f) = \nu(f)
\end{equation}
whenever \( f \) is \( C^\infty \) on a neighborhood of \( m \) and \( \nu \) is a tangent vector of any
order at \( m \).

Finally, just as there are natural mappings of tangent vectors and differentials
associated with a differentiable map \( \varphi: M \to N \), so are there linear
mappings
\begin{equation}
\begin{align*}
\delta^k \varphi: M_m^k & \to N_m^k, \\
\delta^k \varphi: N_m^k & \to M_m^k
\end{align*}
\end{equation}
defined by
\begin{equation}
\begin{align*}
\delta^k \varphi (v)(g) &= \nu (g \circ \varphi), \\
\delta^k \varphi (d^g \varphi) &= d^g (\varphi \circ \varphi)
\end{align*}
\end{equation}
whenever \( v \in M_m^k \) and \( g \) is a \( C^\infty \) function on a neighborhood of \( \varphi (m) \).
It is easily checked that (14) does indeed define the mappings (13) and that
the mappings \( \delta^k \varphi \) and \( \delta^k \varphi \) are dual.

Our definition of a first order tangent vector in this section agrees with
Definition 1.14 in view of Lemma 1.16. Moreover, we have seen three
interpretations of the first order differential \( df \) of a function \( f \); the
interpretation (13) agrees with our original definition 1.22(1), the interpretation
(6) agrees with 1.22(6), and we have the additional interpretation (1).

**SUBMANIFOLDS, DIFFEOMORPHISMS, AND THE
INVERSE FUNCTION THEOREM**

**1.27 Definitions**

Let \( \varphi: M \to N \) be \( C^\infty \).

(a) \( \varphi \) is an immersion if \( d\varphi_m \) is non-singular for each \( m \in M \).

(b) The pair \( (M, \varphi) \) is a submanifold of \( N \) if \( \varphi \) is a one-to-one immersion.

(c) \( \varphi \) is an imbedding if \( \varphi \) is a one-to-one immersion which is also a
homeomorphism into; that is, \( \varphi \) is open as a map into \( \varphi(M) \) with the
relative topology.

(d) \( \varphi \) is a diffeomorphism if \( \varphi \) maps \( M \) one-to-one onto \( N \) and \( \varphi^{-1} \)
is \( C^\infty \).

**1.28 Remarks on 1.27**

One can, for example, immerse the real
line \( \mathbb{R} \) into the plane, as illustrated in the following figure, so that the first
case is an immersion which is not a submanifold, the second is a submanifold
which is not an imbedding, and the third is an imbedding.

Observe that if \((U, \varphi)\) is a coordinate system, then \( \varphi: U \to \varphi(U) \) is a
diffeomorphism.

The composition of diffeomorphisms is again a diffeomorphism. Thus
the relation of being diffeomorphic is an equivalence relation on the collection
of differentiable manifolds. It is quite possible for a locally Euclidean
space to possess distinct differentiable structures which are diffeomorphic.
See Exercise 2.) In a remarkable paper, Milnor showed the existence of
locally Euclidean spaces \( (S^2 \) is an example) which possess non-diffeomorphic
differentiable structures [19]. There are also locally Euclidean spaces which
possess no differentiable structures at all [14].

If \( \varphi \) is a diffeomorphism, then \( d\varphi_m \) is an isomorphism since both \( (d\varphi \circ d\varphi^{-1}) \mid_{\varphi(m)} \) and \( (d\varphi^{-1} \circ d\varphi) \mid_{m} \) are the identity transformations.
The inverse function theorem gives us a local converse of this—whenever \( d\varphi_m \)
is an isomorphism, \( \varphi \) is a diffeomorphism on a neighborhood of \( m \). Before
we recall the precise statement of the inverse function theorem, we give a
definition which will be needed in the corollaries.

**1.29 Definition**

A set \( y_1, \ldots, y_l \) of \( C^\infty \) functions defined on some
neighborhood of \( m \) in \( M \) is called an independent set at \( m \) if the differentials
\( dy_1, \ldots, dy_l \) form an independent set in \( M_m^l \).

**1.30 Inverse Function Theorem**

Let \( U \subset \mathbb{R}^d \) be open, and let
\( f: U \to \mathbb{R}^d \) be \( C^\infty \). If the Jacobian matrix
\[
\frac{\partial f}{\partial y_j}(x)
\]
is non-singular at \( x \in U \), then there exists an open set \( V \) with \( x \in V \subset U \)
such that \( f \mid V \) maps \( V \) one-to-one onto the open set \( f(V) \), and \( f^{-1} \mid f(V) \) is \( C^\infty \).

This is one of the results we shall assume from advanced calculus. For
a proof, we refer the reader, for example, to [31] or [6].
Corollary (a) **Assume that** \( \psi: M \to N \) is \( C^\infty \), that \( m \in M \), and that \( d\psi: M_m \to N_{\psi(m)} \) is an isomorphism. Then there is a neighborhood \( U \) of \( m \) such that \( \psi: U \to \psi(U) \) is a diffeomorphism onto the open set \( \psi(U) \) in \( N \).

**Proof.** Observe that \( \dim M = \dim N \), say \( d \). Choose coordinate systems \( (V, \varphi) \) about \( m \) and \( (W, \tau) \) about \( \psi(m) \) with \( \psi(V) \subset W \). Let \( \varphi(m) = p \) and \( \tau(\varphi(m)) = q \). The differential of the map \( \tau \circ \varphi^{-1} \circ \psi \) is non-singular at \( p \). Thus the inverse function theorem yields a diffeomorphism \( \alpha: U \to \alpha(U) \) on a neighborhood \( U \) of \( p \) with \( \alpha(U) \subset \tau(V) \). Then \( \tau^{-1} \circ \alpha \circ \varphi \) is the required diffeomorphism on the neighborhood \( U = \varphi^{-1}(U) \) of \( m \).

**Corollary (b) Suppose that** \( \dim M = d \) and that \( y_1, \ldots, y_d \) is an independent set of functions at \( m_0 \in M \). Then the functions \( y_1, \ldots, y_d \) form a coordinate system on a neighborhood of \( m_0 \).

**Proof.** Suppose that the \( y_i \) are defined on the open set \( U \) containing \( m_0 \). Define \( \psi: U \to \mathbb{R}^d \) by

\[
\psi(m) = (y_1(m), \ldots, y_d(m)) \quad (m \in U).
\]

Then \( \psi \) is \( C^\infty \). Now \( \delta \psi \) is an isomorphism on \( (\mathbb{R}^d, \psi(m), \psi(V)) \) since

\[
\delta \psi(\delta y_i) = \delta r \circ \psi = dy_i
\]

which implies that \( \delta \psi \) takes a basis to a basis. Consequently, the differential \( \delta \psi \) (which is the dual of \( \delta \psi \)) is an isomorphism. So the inverse function theorem implies that \( \psi \) is a diffeomorphism on a neighborhood \( V \subset U \) of \( m_0 \), and consequently the functions \( y_1, \ldots, y_d \) yield a coordinate system when restricted to \( V \).

**Corollary (c) Suppose that** \( \dim M = d \) and that \( y_1, \ldots, y_l \), with \( l < d \), is an independent set of functions at \( m \). Then they form part of a coordinate system on a neighborhood of \( m \).

**Proof.** Let \( (U, x_1, \ldots, x_d) \) be a coordinate system about \( m \). Then \( \{dx_1, \ldots, dx_d\} \) spans \( M_m^* \). Choose \( d-l \) of the \( x_i \) so that \( \{dx_1, \ldots, dx_l, dx_{l+1}, \ldots, dx_d\} \) is a basis of \( M_m^* \). Then apply Corollary (b).

**Corollary (d) Let** \( \psi: M \to N \) be \( C^\infty \), and assume that \( d\psi: M_m \to N_{\psi(m)} \) is surjective. Let \( x_1, \ldots, x_l \) form a coordinate system on some neighborhood of \( \psi(m) \). Then \( x_i \circ \psi, \ldots, x_l \circ \psi \) form part of a coordinate system on some neighborhood of \( m \).

**Proof.** The fact that \( d\psi \) is surjective implies that the dual map \( d\psi^* \) is injective. Thus the functions \( \{x_i \circ \psi : i = 1, \ldots, l\} \) are independent at \( m \) since \( \delta \psi(\delta x_i) = d(x_i \circ \psi) \). The claim now follows from Corollary (c).

**Corollary (e) Suppose that** \( y_1, \ldots, y_k \) is a set of \( C^\infty \) functions on a neighborhood of \( m \) such that their differentials span \( M_m^* \). Then a subset of the \( y_i \) forms a coordinate system on a neighborhood of \( m \).

**Proof.** Simply choose a subset whose differentials form a basis of \( M_m^* \), and apply Corollary (b).

**Corollary (f) Let** \( \psi: M \to N \) be \( C^\infty \), and assume that \( d\psi: M_m \to N_{\psi(m)} \) is injective. Let \( x_1, \ldots, x_k \) form a coordinate system on a neighborhood of \( \psi(m) \). Then a subset of the \( x_i \circ \psi \) forms a coordinate system on a neighborhood of \( m \). In particular, \( \psi \) is one-to-one on a neighborhood of \( m \).

**Proof.** The fact that \( d\psi \) is injective implies that \( d\psi^* \) is surjective. This implies that \( d(x_i \circ \psi) = d(x_i \circ \psi) : i = 1, \ldots, k \) spans \( M_m^* \). This corollary then follows from Corollary (e).

1.31 The situation often arises that one has a \( C^\infty \) mapping, say \( \psi \), of a manifold \( N \) into a manifold \( M \) factoring through a submanifold \( (\mathbb{R}^d, \varphi) \) of \( M \). That is, \( \psi(N) \subset \varphi(P) \), whence there is a uniquely defined mapping \( \psi \) of \( N \) into \( P \) such that \( \varphi \circ \psi = \varphi \). The problem is: When is \( \psi \) of class \( C^\infty \)? This is certainly not always the case. As an example, let \( N \) and \( P \) both be the real line, and let \( M \) be the plane. Let \((\mathbb{R}, \psi)\) and \((\mathbb{R}, \varphi)\) both be figure-8 submanifolds with precisely the same image sets, but with the difference that as \( t \to \pm \infty \), \( \psi(t) \) approaches the intersection along the horizontal direction, but \( \varphi(t) \) approaches along the vertical. Suppose also that \( \psi(0) = \varphi(0) = 0 \). Then \( \psi \) is not even continuous since \( \psi^{-1}(-1,1) \) consists of the origin plus two open sets of the form \( (a, +\infty), (-\infty, -a) \) for some \( a > 0 \).
1.32 Theorem  Suppose that \( \psi : N \to M \) is \( C^\infty \), that \( (P, \varphi) \) is a submanifold of \( M \), and that \( \psi \) factors through \( (P, \varphi) \), that is, \( \psi(N) \subseteq \varphi(P) \). Since \( \varphi \) is injective, there is a unique mapping \( \psi_0 \) of \( N \) into \( P \) such that \( \varphi \circ \psi_0 = \psi \).

\[
\begin{array}{c}
N \xrightarrow{\psi} M \\
\downarrow \quad \psi_0 \quad \downarrow \varphi \\
\downarrow \quad p \quad \downarrow \end{array}
\]

(a) \( \psi_0 \) is \( C^\infty \) if it is continuous.

(b) \( \psi_0 \) is continuous if \( \varphi \) is an imbedding.

Another important case in which \( \psi_0 \) is continuous occurs when \( (P, \varphi) \) is an integral manifold of an involutive distribution on \( M \), as we shall see in 1.62.

**Proof** Result (b) is obvious. So assume that \( \psi_0 \) is continuous. We prove that it is \( C^\infty \). It suffices to show that \( P \) can be covered by coordinate systems \((U, \tau)\) such that the map \( \tau \circ \psi_0 \) restricted to the open set \( \varphi^{-1}(U) \) is \( C^\infty \). Let \( p \in P \), and let \((V, \gamma)\) be a coordinate system on a neighborhood of \( \varphi(p) \) in \( M \). Then by Corollary (f) of 1.30 there exists a projection \( \pi \) of \( \mathbb{R}^d \) onto a suitable subspace (obtained by setting certain of the coordinate functions equal to 0) such that the map \( \tau = \pi \circ \gamma \circ \varphi \) yields a coordinate system on a neighborhood of \( U \) of \( p \). Then

\[
\tau \circ \psi_0 \big| \varphi^{-1}(U) = \pi \circ \gamma \circ \varphi \circ \psi_0 \big| \varphi^{-1}(U) = \pi \circ \gamma \circ \varphi \big| \varphi^{-1}(U),
\]

which is \( C^\infty \).

1.33 Further Remarks on Submanifolds  Submanifolds \((N_1, \varphi_1)\) and \((N_2, \varphi_2)\) of \( M \) will be called equivalent if there exists a diffeomorphism \( \alpha : N_1 \to N_2 \) such that \( \varphi_2 \circ \alpha = \varphi_1 \).

\[
\begin{array}{c}
N_1 \xrightarrow{\varphi_1} M \\
\downarrow \quad \alpha \quad \downarrow \\
N_2 \xrightarrow{\varphi_2} M
\end{array}
\]

This is an equivalence relation on the collection of all submanifolds of \( M \). Each equivalence class \( \xi \) has a unique representative of the form \((A,i)\) where \( A \) is a subset of \( M \) with a manifold structure such that the inclusion map \( i: A \to M \) is a \( C^\infty \) immersion. Namely, if \((N, \varphi)\) is any representative of \( \xi \), then the subset \( A \) of \( M \) must be \( \varphi(N) \). We induce a manifold structure on \( A \) by requiring \( \varphi: N \to A \) to be a diffeomorphism. With this manifold structure, \((A,i)\) is a submanifold of \( M \) equivalent to \((N, \varphi)\). This is the only manifold structure on \( A \) with the property that \((A,i)\) is equivalent to \((N, \varphi)\); thus this is the unique such representative of \( \xi \).

The conclusion of some theorems in the following sections state that there exist unique submanifolds satisfying certain conditions. Uniqueness means up to equivalence as defined above. In particular, if the submanifolds of \( M \) are viewed as subsets \( A \subset M \) with manifold structures for which the inclusion maps are \( C^\infty \) immersions, then uniqueness means unique subset with unique second countable locally Euclidean topology and unique differentiable structure.

In the case of a submanifold \((A,i)\) of \( M \) where \( i \) is the inclusion map, we shall often drop the \( i \) and simply speak of the submanifold \( A \subset M \).

Let \( A \) be a subset of \( M \). Then generally there is not a unique manifold structure on \( A \) such that \((A,i)\) is a submanifold of \( M \), if there is one at all. For example, the diagrams in 1.31 illustrate two distinct manifold structures on the figure-8 in the plane, each of which makes the figure-8 together with the inclusion map a submanifold of \( \mathbb{R}^2 \). However, we have the following two uniqueness theorems which involve conditions on the topology on \( A \).

(a) Let \( M \) be a differentiable manifold and \( A \) a subset of \( M \). Fix a topology on \( A \). Then there is at most one differentiable structure on \( A \) such that \((A,i)\) is a submanifold of \( M \), where \( i \) is the inclusion map.

(b) Again let \( A \) be a subset of \( M \). If in the relative topology, \( A \) has a differentiable structure such that \((A,i)\) is a submanifold of \( M \), then \( A \) has a unique manifold structure (that is, unique second countable locally Euclidean topology together with a unique differentiable structure) such that \((A,i)\) is a submanifold of \( M \).

We leave these to the reader as exercises. Result (a) follows from an application of Theorem 1.32. Result (b) depends strongly on our assumption that manifolds are second countable, and for its proof you will need to use the proposition in Exercise 6 in addition to Theorem 1.32.

1.34 Slices  Suppose that \((U, \varphi)\) is a coordinate system on \( M \) with coordinate functions \( x_1, \ldots, x_d \), and that \( c \) is an integer, \( 0 \leq c \leq d \). Let \( a \in \varphi(U) \), and let

\( S = \{ q \in U: x_i(q) = r_i(a), i = c + 1, \ldots, d \} \).

The subspace \( S \) of \( M \) together with the coordinate system

\( \{ x_j | S: j = 1, \ldots, c \} \)

forms a manifold which is a submanifold of \( M \) called a slice of the coordinate system \((U, \varphi)\).
1.35 Proposition Let \( \psi : M^e \to N^d \) be an immersion, and let \( m \in M \). Then there exists a cubic-centered coordinate system \((V, \varphi)\) about \( \psi(m) \) and a neighborhood \( U \) of \( m \) such that \( \psi \mid U \) is 1:1 and \( \psi(U) \) is a slice of \((V, \varphi)\).

Proof Let \((W, \tau)\) be a centered coordinate system about \( \psi(m) \) with coordinate functions \( y_1, \ldots, y_d \). By Corollary (f) of 1.30 we can renumber the coordinate functions so that

\[
\tau = \pi \circ \tau \circ \psi
\]

is a coordinate map on a neighborhood \( V' \) of \( m \) where \( \pi : \mathbb{R}^d \to \mathbb{R}^c \) is projection on the first \( c \) coordinates. Define functions \( \{x_i\} \) on \((\pi \circ \tau)^{-1}(\tau(V'))\) by setting

\[
x_i = \begin{cases} y_i & (i = 1, \ldots, c) \\ y_i - y_i \circ \psi \circ \tau^{-1} \circ \pi & (i = c + 1, \ldots, d) \end{cases}
\]

The functions \( \{x_i\} \) are independent at \( \psi(m) \), since at \( \psi(m) \),

\[
dx_i = \begin{cases} dy_i & (i = 1, \ldots, c) \\ dy_i + \sum_{j=1}^{d} a_{ij} dy_j & (i = c + 1, \ldots, d) \end{cases}
\]

for some constants \( a_{ij} \). By Corollary (b) of 1.30 the \( \{x_i\} \) form a coordinate system on a neighborhood of \( \psi(m) \). Let \( V' \) be a neighborhood of \( \psi(m) \) on which the \( x_1, \ldots, x_d \) form a cubic coordinate system. Denote the corresponding coordinate map by \( \varphi \). Let \( U = \psi^{-1}(V) \cap V' \). Then \( U \) and \( (V, \varphi) \) are the required neighborhood and coordinate system.

We emphasize that this proposition only says that there is a neighborhood \( U \) of \( m \) such that \( \psi(U) \) is a slice of the coordinate system \((V, \varphi)\). Even if \((M, \psi)\) is a submanifold of \( N \), it may well be that \( \psi(M) \cap V \) is far from being a slice or even a union of slices. For an example, consider again the figure-8 submanifold of the plane:

However, in the case that \((M, \psi)\) is an imbedded submanifold, the coordinate system \((V, \varphi)\) can be chosen so that all of \( \psi(M) \cap V \) is a single slice of \( V \).

Let us now consider the question of the extent to which the set of \( C^\infty \) functions on a manifold determines the set of \( C^\infty \) functions on a submanifold. Let \((M, \psi)\) be a submanifold of \( N \). Then, of course, if \( f \in C^\infty(N) \), then \( f \mid M \) is a \( C^\infty \) function on \( M \). (More precisely, \( f \circ \psi \) is a \( C^\infty \) function on \( M \).) In general, however, the converse does not hold: that is, not all \( C^\infty \) functions on \( M \) arise as the restrictions to \( M \) of \( C^\infty \) functions on \( N \). For the converse to hold, it is necessary and sufficient to assume that \( \psi \) is an imbedding and that \( \psi(M) \) is closed. We prove the sufficiency in the following proposition, and leave the necessity as Exercise 11 below.

1.36 Proposition Let \( \psi : M \to N \) be an imbedding such that \( \psi(M) \) is closed in \( N \). If \( g \in C^\infty(M) \), then there exists \( f \in C^\infty(N) \) such that \( f \circ \psi = g \).

To simplify notation, we shall suppress the map \( \psi \) and consider \( M \subset N \).

Proof For each point \( p \in M \) there exists an open set \( O_p \) in \( N \) containing \( p \) and an extension of \( g \) from \( O_p \cap M \) to a \( C^\infty \) function \( \tilde{g}_p \) on \( O_p \). One simply has to take \( O_p \) to be a cubic-centered coordinate neighborhood of \( p \) for which \( M \cap O_p \) is a single slice, and then define \( \tilde{g}_p \) to be the composition of the natural projection of \( O_p \) onto the slice followed by \( g \). The collection \( \{O_p : p \in M\} \) together with \( N - M \) forms an open cover of \( N \). By Theorem 1.11, there exists a partition of unity \( \{\varphi_j\} \), with \( f = 1, 2, \ldots \), subordinate to this cover. Take the subsequence (which we shall continue to denote by \( \{\varphi_j\} \)) such that \( \text{supp } \varphi_j \neq \emptyset \). For each such \( j \), we can choose a point \( p_j \) such that \( \text{supp } \varphi_j = O_{p_j} \). Then \( f = \sum_j \varphi_j \tilde{g}_{p_j} \) is a \( C^\infty \) function on \( N \), and \( f \mid M = g \).