1) A particle is moving in the $xy$-plane. At time $t$ seconds, the position $(x, y)$ of the particle, in meters, is given by

$$x = 8 - t^2, \quad y = t/4.$$

a) (8 points) Sketch a portion of the $xy$-plane, using a scale of 1 on the $x$-axis and a scale of 1/4 on the $y$-axis. On your sketch, mark the position of the particle at times $t = 0, 1, 2, 3, 4$ (indicate the times at these points). Connect the points by a smooth path, and indicate with arrows the direction in which the particle is moving.

![Sketch of particle motion](image)

b) (4 points) Eliminate the $t$ in the equations above to find an equation involving $x$ and $y$ for the path of the particle.

$$x = 8 - t^2, \quad t = 4y,$$

$$x = 8 - (4y)^2.$$

2)

a) (5 points) Give some parameterization, using $t$ for the parameter, for the circle of radius 3 centered at the origin (in the $xy$-plane). Your answer should be of the form $x$ equals some function of $t$ and $y$ equals some function of $t$.

$$x = 3 \cos t, \quad y = 3 \sin t.$$

b) (3 points) Using the parameterization that you gave in a), what point $(x, y)$ do you get when $t = 0$ and, as $t$ increases, does the point given by the parameterization move clockwise or counterclockwise around the circle?

At $t = 0$, $x = 3$ and $y = 0$. 

**Counterclockwise.**
3) (6 points each)
a) A particle is moving along the x-axis. Suppose that its position \( x = x(t) \), in meters, at time \( t \) seconds is given by \( x = x(t) = t^2 \). What is the average velocity of the particle between times \( t = 3 \) and \( t = 5 \)? Be sure to include units.

\[
\text{Avg. vel.} = \frac{\Delta x}{\Delta t} = \frac{x(5)-x(3)}{5-3} \quad \text{m/s}
\]

\[
= \frac{5^2-3^2}{5-3} \quad \text{m/s} = \frac{16}{2} \quad \text{m/s} = 8 \quad \text{m/s}.
\]

b) A different particle is moving along the x-axis. Suppose that its velocity \( v = v(t) \), in m/s, at time \( t \) seconds is given by \( v = v(t) = t^2 \). What is the average acceleration of the particle between times \( t = 3 \) and \( t = 5 \)? Be sure to include units.

\[
\text{Avg. accel.} = \frac{\Delta v}{\Delta t} = \frac{v(5)-v(3)}{5-3} \quad \text{m/s}^2
\]

\[
= \frac{5^2-3^2}{5-3} \quad \text{m/s}^2 = 8 \quad \text{m/s}^2.
\]

c) A steel rod of variable density lies along the x-axis, with one end at \( x = 0 \) and the other at \( x = 12 \) meters. The total mass of the rod, in kg, between 0 and \( x \) meters is given by \( m(x) = x^2 \). What is the average rate of change of the mass of the rod, with respect to \( x \), between \( x = 3 \) and \( x = 5 \)? Be sure to include units.

\[
\frac{\Delta m}{\Delta x} = \frac{m(5)-m(3)}{5-3} \quad \text{kg/m} = \frac{5^2-3^2}{5-3} \quad \text{kg/m}
\]

\[
= 8 \quad \text{kg/m}.
\]

4) (13 points) Let \( P(v) = 5v^2 + 3v \). Find \( P'(-1) \).

\[
P'(-1) = \lim_{h \to 0} \frac{P(-1+h)-P(-1)}{h} = \]

\[
\lim_{h \to 0} \frac{5(-1+h)^2 + 3(-1+h) - (5(-1)^2 + 3(-1))}{h} = \]

\[
\lim_{h \to 0} \frac{5(-1-2h+h^2) - 3 + 3h - 2}{h} = \]

\[
\lim_{h \to 0} \frac{5-10h+5h^2 - 5 + 3h}{h} = \lim_{h \to 0} \frac{-7h + 5h^2}{h} = \]

\[
\lim_{h \to 0} (-7 + 5h) = -7.
\]
5) Let \( y = f(x) = \frac{3}{x} \).

a) (10 points) Find \( f'(x) \), using the definition as a limit. You will get zero credit for using memorized formulas.

\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h} = \lim_{h \to 0} \frac{3x - 3(x+h)}{(x+h)x} \cdot \frac{1}{h} = \lim_{h \to 0} \frac{-3h}{(x+h)x} = \lim_{h \to 0} \frac{-3}{(x+h)x} = \frac{-3}{x^2}.
\]

b) (6 points) Use your answer to part a) to find an equation for the tangent line to the graph of \( y = \frac{3}{x} \) at the point on the graph where \( x = 2 \).

\[
\text{slope} = f'(2) = -\frac{3}{4}.
\]

\((x_1, y_1) = (2, \frac{3}{2})\). \quad y - y_1 = m(x - x_1).

\[
y - \frac{3}{2} = -\frac{3}{4} (x - 2).
\]

6) (3 points each)

a) If \( x = x(t) \) is the position, in meters, of a particle moving along the \( x \)-axis, at time \( t \) seconds, what is (i.e., what does the average person in the street call) \( x''(t) = \frac{d^2x}{dt^2} \)?

**Acceleration.**

b) What are the units of \( x''? \)

\[
\frac{(\text{meters})}{\text{seconds}^2} \text{ or } \frac{\text{meters}}{\text{seconds}^2}.
\]
7) (5 points each) Consider the graph below of \( y = f(x) \)

![Graph of \( y = f(x) \)](image)

and use it to estimate the \( x \) values specified below.

a) For what \( x \) values is \( f'(x) > 0? \)

\[ x < -4 \text{ and } x > 1. \]

b) For what \( x \) values is \( f''(x) < 0? \)

\[ x < -2 \]

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8) (12 points) Write a short essay explaining why the derivative is the instantaneous rate of change, i.e., explain why we define the derivative the way that we do.

It may help you to think about, and maybe refer to, our original discussion about measuring the velocity of a car, at a particular time, by using measurements from outside the car. It may also help you to think about how you would explain, to a friend who has not taken Calculus, why one would define anything using the “strange” limit definition that we use.

Your essay should continue onto the next page, which is blank.

**Essay:** Suppose \( y = f(x) \). The average rate of change of \( f \) with respect to \( x \) between \( x = a \) and \( x = b \), where \( a \neq b \), is

\[ \frac{f(b) - f(a)}{b - a}. \]

How would one
approximate the instantaneous rate of change of $f$ with respect to $x$ at $x = a$? By calculating the average rate of change $\frac{f(b) - f(a)}{b - a}$, where $b$ is very very close to $a$. So, we define the derivative $f'(a)$ to be what $\frac{f(b) - f(a)}{b - a}$ gets closer and closer to as $b$ gets closer and closer to $a$, i.e., we define

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a},$$

which makes it the instantaneous rate of change.